Alejandro Jiménez | Complex Analysis Exercises

(7.1) $\gamma_1 : [0,1] \longrightarrow \mathbb{C}$ $\qquad \gamma_1(0) = 1\exp(0) = 1.$

$\gamma_1(t) = (1+t^2)\exp(i3\pi\sqrt{t})$. $\qquad \gamma_1(1) = 2\underbrace{\exp(i3\pi)}_{-1} = -2.$

$\frac{1}{z^2}$ has a primitive $-\frac{1}{z}$. Then we can apply FTC:

$$\int_{\gamma_1} \frac{dz}{z^2} = \frac{-1}{-2} - \frac{-1}{1} = \frac{3}{2}.$$

$\gamma_2 : [0,1] \longrightarrow \mathbb{C}$ $\qquad \gamma_2(0) = 1+1 = 2$

$\gamma_2(t) = 1 + \exp(2\pi i t)$ $\qquad \gamma_2(1) = 1 + \underbrace{\exp(2\pi i)}_{1} = 2.$

$\Rightarrow \gamma_2(0) = \gamma_2(1)$ so γ_2 is closed.

And $\sin(z^a)$ is a composition of analytic functions, thus analytic. So, by the Cauchy-Goursat theorem, as γ_2 is closed: $\displaystyle\int_{\gamma_2} \sin(z^a)\,dz = 0.$

(7.2) $f(z) = z^2\sin(z)$.

$$\int z^2\sin(z) = \begin{bmatrix} u = z^2 & dv = \sin z \\ du = 2z & v = -\cos z \end{bmatrix} = -z^2\cos z + 2\int z\cos z\,dz.$$

$$\int z\cos z\,dz = \begin{bmatrix} u = z & dv = \cos z \\ du = 1 & v = \sin z \end{bmatrix} = z\sin z - \int \sin z\,dz =$$

$$= z\sin z + \cos z.$$

$$\Rightarrow \int z^2\sin(z) = -z^2\cos z + 2z\sin z + 2\cos z + C \quad (C \text{ constant}).$$

So $F(z) = -z^2\cos z + 2z\sin z + 2\cos z$ is a primitive of F.

Then by FTC: $\displaystyle\int_\gamma f(z)\,dz = F(i) - F(0) =$

$= \cos i + 2i\sin i + 2\cos i - 2\cos 0 = \underline{3\cos(i) + 2i\sin(i) - 2}.$

(13) $\eta \in \mathbb{C},\ |\eta| = 1,\ a \in \mathbb{C},\ |a| < 1.\quad f: \mathbb{D} \to \mathbb{D}$

$$\left(f(z) = \eta\,\frac{z+a}{1+\bar{a}z} \right)$$

$$f'(z) = \eta\,\frac{(1+\bar{a}z) + (z+a)\bar{a}}{(1+\bar{a}z)^2} =$$

$$= \eta\,\frac{1-|a|^2}{(1+\bar{a}z)^2} \Rightarrow |f'(z)| = |\eta|\left|\frac{1-|a|^2}{(1+\bar{a}z)^2}\right| \overset{|\eta|=1}{=} \frac{1-|a|^2}{|1+\bar{a}z|^2}.$$

$|a| < 1 \Rightarrow |a|^2 < 1$

$$\text{Now } \frac{|f'(z)|}{1-|f(z)|^2} = \frac{1-|a|^2}{|1+\bar{a}z|^2\left(1 - \underbrace{|\eta^2|}_{=1}\left|\frac{z+a}{1+\bar{a}z}\right|^2\right)} = \frac{1-|a|^2}{\underbrace{|1+\bar{a}z|^2}_{(1+\bar{a}z)(1+a\bar{z})} - \underbrace{|z+a|^2}_{(z+a)(\bar{z}+\bar{a})}} =$$

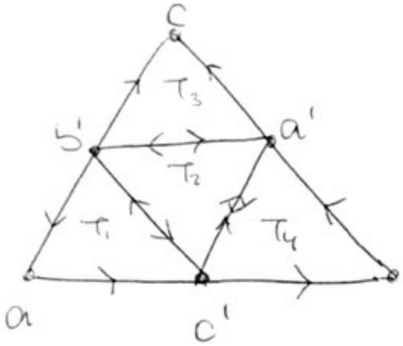
$$= \frac{1-|a|^2}{1+|a|^2|z|^2 - |z|^2 - |a|^2} = \frac{\cancel{1-|a|^2}}{(\cancel{1-|a|^2})(1-|z|^2)} = \underline{\frac{1}{1-|z|^2}}.$$

$$\text{Then: } l_{hyp}(f\circ\gamma) = \int_{f\circ\gamma} \frac{1}{1-|z|^2}\,|dz| = \int_{(f\circ\gamma)_1}^{(f\circ\gamma)_2} \frac{1}{1-|f(\gamma(t))|^2}\,\underbrace{|(f(\gamma(t)))'|}_{|f'(\gamma(t))||\gamma'(t)|}\,dt =$$

$$= \int_{(f\circ\gamma)_1}^{(f\circ\gamma)_2} \underbrace{\frac{|f'(\gamma(t))|}{1-|f(\gamma(t))|^2}}_{=\frac{1}{1-|\gamma(t)|^2} \text{ because } \gamma(t)\in\mathbb{D}} |\gamma'(t)|\,dt = \int_{\gamma_1}^{\gamma_2} \frac{1}{1-|\gamma(t)|^2}\,|\gamma'(t)| = \int_\gamma \frac{1}{1-|z|^2}\,|dz| =$$

$$= \underline{l_{hyp}(\gamma)}.$$

(7.4) The triangle $T \subset D$ has vertices a, b, c. Now let a', b', c' be the midpoints of segments $[b,c]$, $[c,a]$, $[a,b]$.



Then we divide the triangle T in 4 triangles T_1, T_2, T_3, T_4, and the boundary of all of them is oriented clockwise. Then:

$$J = \int_{\partial T} f(z)\,dz = \sum_{k=1}^{4} \int_{\partial T_k} f(z)\,dz;$$ by the triangle inequality for some $T_{k'}$ (k' fixed):

$$|J| \le 4 \left| \int_{\partial T_{k'}} f(z)\,dz \right|$$ We denote this T_k with T' and apply the same process to T' so that for $J_1 = \int_{\partial T'} f(z)\,dz$ we find a triangle T^2. We iterate the process indefinetly, and then we obtain a sequence of nested triangles:

$T \supset T' \supset T^2 \supset \dots$ with the property $|J| \le 4|J_1| \le 4^2|J_2| \le \dots$

If $J_j := \int_{\partial T^i} f(z)\,dz \Rightarrow |J_j| \le 4|J_{j+1}|$ Hence $|J| \le 4^i |J_i|$.

We also have that the leght of ∂T^i is 2^{-i} times the lenght of ∂T. $\Rightarrow \lim_{n \to \infty} \operatorname{diam}(T^i) = 0$ because $\operatorname{diam}(T^i) \le \text{lenght}(\partial T^i)$. Now by the lemma of the nested sets: we know $\bigcap_i T^i$ contains a point z_0.

f is complex diff. in that point, so for $\varepsilon > 0$ arbitrarily chosen and fixed there exists $\delta > 0$ with

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0| \text{ for } z \in \mathbb{D}(z_0, \delta).$$

We chose now a j such that $T^j \subset \mathbb{D}(z_0, \delta)$. Then for this j: $|z - z_0| \leq 2^{-j} \operatorname{length}(\partial T)$. So:

$$\int_{\partial T^j} f(z)dz = \int_{\partial T^j} f(z) - f(z_0) - f'(z_0)(z - z_0)dz.$$

$\int_{\partial T^j} f(z_0)dz = 0$. $\int_{\partial T^j} z dz = 0$ (z and the constant function have primitives)

$$\Rightarrow |J_j| \leq \int_{\partial T^j} \varepsilon |z - z_0| dz = \varepsilon |z - z_0| \operatorname{length}(\partial T^j) = \varepsilon (2^{-j} \operatorname{length}(\partial T))^2$$

Then $|J| \leq 4^n \left| \int_{\partial T^j} f(z)dz \right| \leq \varepsilon (\operatorname{lenght}(\partial T))^2$ for any $\varepsilon > 0$.

Then $\underline{J = 0}$.

(7.5.) f cont. differentiable (complex). Then:

$f(z) = U(z) + iV(z)$ for u, v real functions.

Then: $\displaystyle\int_{\partial T} f(z)\,dz = \int_{\partial T} (u + iv)(dx + idy) = \int_{\partial T} (U + iV)dx + (iU - V)dy$

$\overset{\text{Green's theorem}}{=} \displaystyle\iint_{\operatorname{int} T} \left(\frac{\partial(U + iV)}{\partial x} + \frac{\partial(iU - V)}{\partial y} \right) dx dy$. As f is comp. diff, then u and v satisfy the Cauchy-Riemann equations.

Then: $\int_{\partial T} f(z)dz = \int_{\text{int } T} \Big(\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{= \frac{\partial v}{\partial y}} + i \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{-\frac{\partial u}{\partial y}} + i \cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial v}{\partial y}} \Big) dx\,dy = \underline{\underline{0}}.$