# Fourier Analysis II

Spring 2020

Homework 1

Exercise session: Mon 23 March; Stefanos Lappas, stefanos.lappas@helsinki.fi.

**1.** Compute the Fourier transform of the characteristic function $\chi_{[-a,a]}$ (here $a > 0$).

*Proof.* Let $f(x) = \chi_{[-a,a]}(x), a > 0$ and $\xi \neq 0$. We compute

$$\widehat{f}(\xi) = \int_{-a}^{a} e^{-i\xi x} dx = \frac{e^{-i\xi a} - e^{i\xi a}}{-i\xi} = \frac{2\sin(a\xi)}{\xi}.$$

Let $\xi = 0$. Then, we have

$$\widehat{f}(0) = \int_{-a}^{a} 1 dx = 2a.$$

Since $\widehat{f}$ is continuous, observe that $\lim_{\xi \to 0} \frac{2\sin(a\xi)}{\xi} = 2a = \widehat{f}(0)$. Thus,

$$\widehat{f}(\xi) = \begin{cases} \frac{2\sin(a\xi)}{\xi}, & \text{if } \xi \neq 0 \\ 2a, & \text{if } \xi = 0. \end{cases}$$

□

**2.** **(i)** Compute the convolution $\chi_{[-a,a]} * \chi_{[-a,a]}$.
**(ii)** Compute the Fourier transform (in one dimension) of the function $g(x) = \max(0, 1 - |x|)$.

*Proof.* **(i)** A straightforward computation shows that

$$\begin{aligned}
\chi_{[-a,a]} * \chi_{[-a,a]}(x) &= \int_{-\infty}^{\infty} \chi_{[-a,a]}(x-y)\chi_{[-a,a]}(y)dy \\
&= \int_{-\infty}^{\infty} \chi_{[x-a,x+a]}(y)\chi_{[-a,a]}(y)dy \\
&= \int_{-\infty}^{\infty} \chi_{[x-a,x+a]\cap[-a,a]}(y)dy \\
&= \begin{cases} 2a - x, & \text{if } x \geq 0 \text{ and } y \in [x-a, a] \\ 2a + x, & \text{if } x < 0 \text{ and } y \in [-a, x+a] \\ 0, & \text{if } |x| \geq 2a \end{cases} \\
&= \max(0, 2a - |x|).
\end{aligned}$$

**(ii)** By **(i)** for $a = 1/2$ we see that $g(x) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(x)$. Using Exercise 1 and the fact that the Fourier transform of the convolution is the product of the Fourier transforms, we get

$$\widehat{g}(\xi) = (\widehat{\chi}_{[-1/2,1/2]}(\xi))^2 = \begin{cases} \frac{4\sin^2(\xi/2)}{\xi^2}, & \text{if } \xi \neq 0 \\ 1, & \text{if } \xi = 0. \end{cases}$$

□

**3.** Compute the Fourier transform $f : \mathbb{R} \to \mathbb{R}$, where $f(x) := e^{-k|x|}$ (here $k > 0$) : show that

$$\widehat{f}(\xi) = \frac{2k}{k^2 + \xi^2}.$$

*Proof.* This is a pretty straightforward computation

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} e^{-k|x|} dx \\ &= \int_{0}^{\infty} e^{-(i\xi + k)x} dx + \int_{-\infty}^{0} e^{-(i\xi - k)x} dx \\ &= \left[ \frac{1}{-(i\xi + k)} e^{-(i\xi + k)x} \right]_0^{\infty} + \left[ \frac{1}{-(i\xi - k)} e^{-(i\xi - k)x} \right]_{-\infty}^{0} \\ &= \frac{1}{i\xi + k} - \frac{1}{i\xi - k} \\ &= \frac{2k}{k^2 + \xi^2}. \end{aligned}$$

Here we have used the facts that

$$\lim_{x \to \infty} e^{-(i\xi + k)x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} e^{-(i\xi - k)x} = 0,$$

following from $|e^{-(i\xi + k)x}| = e^{-kx}$ and $|e^{-(i\xi - k)x}| = e^{kx}$. □

**4.** **(i)** If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \overline{f(-x)}$, show that $\widehat{g}(\xi) \equiv \overline{\widehat{f}(\xi)}$.

**(ii)** If $f \in L^1(\mathbb{R}^d)$ and $g(x) = \frac{1}{t^d} f\left(\frac{x}{t}\right)$, $t > 0$, show that $\widehat{g}(\xi) \equiv \widehat{f}(t\,\xi)$.

*Proof.* **(i)** We compute simply that

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi x}\overline{f(-x)}dx \\
&= \int_{\mathbb{R}^d} e^{i\xi x}\overline{f(x)}dx \\
&= \int_{\mathbb{R}^d} \overline{e^{-i\xi x}f(x)}dx \\
&= \overline{\int_{\mathbb{R}^d} e^{-i\xi x}f(x)dx} \\
&= \overline{\widehat{f}(\xi)}.
\end{aligned}$$

**(ii)** Using the change of variables $y = x/t = (x_1/t, x_2/t, \ldots, x_n/t)$ whose Jacobian determinant is equal to $1/t^d$, we find that

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi x} f\left(\frac{x}{t}\right)\frac{1}{t^d}dx \\
&= \int_{\mathbb{R}^d} e^{-i\xi ty} f(y)dy \\
&= \int_{\mathbb{R}^d} e^{-it\xi y} f(y)dy \\
&= \widehat{f}(t\xi).
\end{aligned}$$

□

**5.** Suppose that the function $f : \mathbb{R}^d \to \mathbb{C}$ has the form

$$f(x) = f_1(x_1)f_2(x_2)\cdots f_d(x_d), \qquad \forall\, x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where $f_1, \ldots, f_d \in L^1(\mathbb{R}^1)$. Show that then $f \in L^1(\mathbb{R}^d)$ and we have

$$\widehat{f}(\xi) = \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\cdots\widehat{f_d}(\xi_d), \qquad \forall\, \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$

*Proof.* Applying Fubini's theorem for non-negative functions, we see that $\int_{\mathbb{R}^d}|f(x)|dx = \int_{-\infty}^{\infty}|f_1(x_1)|dx_1 \int_{-\infty}^{\infty}|f_2(x_2)|dx_2 \cdots \int_{-\infty}^{\infty}|f_d(x_d)|dx_d < \infty$. As $f \in L^1(\mathbb{R}^d)$, we can now apply Fubini's theorem and compute the Fourier transform. If $d > 1$, we can define a function $g : \mathbb{R}^{d-1} \to \mathbb{C}$ by setting $g(x) = f_1(x_1)f_2(x_2)\cdots f_{d-1}(x_{d-1})$ and denote $\widetilde{\xi} = (\xi_1, \ldots, \xi_{d-1})$. We may then compute

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^d} f(x)e^{-i\xi x}dx \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^{d-1}} g(y)f_d(x_d)e^{-i(\widetilde{\xi}y+\xi_d x_d)}dydx_d \\ &= \int_{-\infty}^{\infty} f_d(x_d)e^{-i\xi_d x_d}\left(\int_{\mathbb{R}^{d-1}} g(y)e^{-i\widetilde{\xi}y}dy\right)dx_d \\ &= \int_{-\infty}^{\infty} f_d(x_d)e^{-i\xi_d x_d}\widehat{g}(\widetilde{\xi})dx_d \\ &= \widehat{f}_d(\xi_d)\widehat{g}(\widetilde{\xi}).\end{aligned}$$

Now $g$ is a function defined on $\mathbb{R}^{d-1}$ so we can use induction to deduce

$$\widehat{f}(\xi) = \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\cdots\widehat{f}_d(\xi_d).$$

□

**6.** Assume that $H \in L^1(\mathbb{R}^d)$ fulfils $H \geq 0$, and $\int_{\mathbb{R}^d} H(x)dx = 1$, together with

$$|H(x)| \leq \frac{C}{(1+|x|)^{d+1}}.$$

For $\epsilon > 0$ let us denote $H_\epsilon(x) := \epsilon^{-d}H(x/\epsilon)$. If $f \in L^1(\mathbb{R}^d)$ is continuous at 0, prove that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} f(x)H_\epsilon(x)dx \ = \ f(0).$$

*Proof.* Let $\eta > 0$ be arbitrary. By continuity of $f$ we can find $\delta > 0$ such that $|f(x) - f(0)| < \eta$ when $|x| < \delta$.
Now as $\int_{\mathbb{R}^d} H_\epsilon(x)dx = \int_{\mathbb{R}^d} H(x)dx = 1$, we have

$$\Big|\int_{\mathbb{R}^d} f(x)H_\epsilon(x)dx - f(0)\Big| = \Big|\int_{\mathbb{R}^d}(f(x)-f(0))H_\epsilon(x)dx\Big|.$$

Split the integral in three parts:

$$\int_{B(0,\delta)}(f(x)-f(0))H_\epsilon(x)dx + \int_{\mathbb{R}^d\setminus B(0,\delta)} f(x)H_\epsilon(x)dx - \int_{\mathbb{R}^d\setminus B(0,\delta)} f(0)H_\epsilon(x)dx.$$

Then in the first integral $|x| < \delta$, so $|f(x) - f(0)| < \eta$. Now we have

$$\Big| \int_{B(0,\delta)} (f(x) - f(0)) H_\epsilon(x) dx \Big| \leq \int_{B(0,\delta)} \eta H_\epsilon(x) dx \leq \eta.$$

We can estimate the second integral as

$$\begin{aligned}
\Big| \int_{\mathbb{R}^d \setminus B(0,\delta)} f(x) H_\epsilon(x) dx \Big| &= \Big| \int_{\mathbb{R}^d \setminus B(0,\delta)} f(x) \frac{H(x/\epsilon)}{\epsilon^d} dx \Big| \\
&\leq \int_{\mathbb{R}^d \setminus B(0,\delta)} |f(x)| \frac{C}{\epsilon^d (1 + |x/\epsilon|)^{d+1)}} dx \\
&\leq \int_{\mathbb{R}^d \setminus B(0,\delta)} |f(x)| \frac{C}{(1 + \delta/\epsilon)^{d+1}} \frac{1}{\epsilon^d} dx \\
&\leq \int_{\mathbb{R}^d \setminus B(0,\delta)} |f(x)| C \frac{\epsilon}{(\epsilon + \delta)^{d+1}} dx \\
&\leq \|f\|_{L^1(\mathbb{R}^d)} C \frac{\epsilon}{\delta^{d+1}} \to 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

For the third integral, we can apply dominated convergence theorem to compute

$$\begin{aligned}
\Big| \int_{\mathbb{R}^d \setminus B(0,\delta)} f(0) H_\epsilon(x) dx \Big| &= \Big| \int_{\mathbb{R}^d \setminus B(0,\delta)} f(0) \frac{H(x/\epsilon)}{\epsilon^d} dx \Big| \\
&= \Big| \int_{\mathbb{R}^d \setminus B(0,\delta/\epsilon)} f(0) H(x) dx \Big| \\
&\leq \int_{\mathbb{R}^d \setminus B(0,\delta/\epsilon)} |f(0)| H(x) dx \to 0, \quad \text{as} \quad \epsilon \to 0.
\end{aligned}$$

We have shown that

$$\limsup_{\epsilon \to 0} \Big| \int_{\mathbb{R}^d} f(x) H_\epsilon(x) dx - f(0) \Big| \leq \eta.$$

As $\eta > 0$ was arbitrary, we have shown that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} f(x) H_\epsilon(x) dx = f(0).$$

□

**7.** Let $a > 0$. Check that the function $H(x) := c_a e^{-a|x|^2}$ with suitable constant $c_a$ satisfies the conditions of the previous exercise. What is the value of $c_a$?

*Proof.* As the real exponential function is always positive, we have that $H(x) \geq 0$ if $c_a \geq 0$.
For the property $\int_{\mathbb{R}^d} H(x)dx = 1$, we compute the integral of $e^{-a|x|^2}$ and then choose $c_a$ appropriately. The integral is slightly modified Gaussian integral, so we use Fubini's theorem:

$$\int_{\mathbb{R}^d} e^{-a|x|^2} dx = \left( \int_{-\infty}^{\infty} e^{-ax^2} dx \right)^d = \sqrt{\frac{\pi}{a}}^d .$$

This means that the integral of $H$ is correct if we choose $c_a = \left(\frac{a}{\pi}\right)^{d/2}$.
It remains to prove for all $x \in \mathbb{R}^d$ the estimate

$$|H(x)| \leq \frac{C}{(1 + |x|)^{d+1}}.$$

For this, observe that it is sufficient to prove for any non-negative real $x$ that

$$c_a e^{-ax^2} \leq \frac{C}{(1 + x)^{d+1}}.$$

Consider the function $g : [0, \infty) \to \mathbb{R}, g(x) = (1 + x)^{d+1} e^{-ax^2}$. Using l'Hospital's rule, we see that $\lim_{x\to\infty} g(x) = 0$. This means that there is a constant $M$ such that $g(x) \leq 1$ for $x > M$. As $g$ is continuous, it has a finite maximum in the interval $[0, M]$, and therefore $g$ is a bounded function.
We have shown that $H$ satisfies the conditions of the previous exercise with constant $c_a = \left(\frac{a}{\pi}\right)^{d/2}$. □

**8.** Prove Leibniz general rule for differentiation of products: if $\alpha \in \mathbb{N}_0^d$ is an arbitrary multi-index and $f, g \in C^\infty(\mathbb{R}^d)$, then

$$\partial^\alpha (fg)(x) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha-\beta} g(x),$$

where $\binom{\alpha}{\beta} := \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}$.

*Proof.* We prove the claim by induction. The case $\alpha = 0$ is trivial. Assume that the claim holds for any $|\alpha| \leq n, n \geq 0$. Then if $|\alpha| = n+1$, we may write $\alpha = \beta + e_i$ for some

$i \in \{1, \dots, d\}$ and $\beta \in \mathbb{N}_0^d$ ($|\beta| = n$). Here $e_i$ is a multi-index with $(e_i)_j = 1$ if $j = i$ and $(e_i)_j = 0$ otherwise. Now using induction hypothesis we get

$$\begin{aligned}
\partial^\alpha(fg)(x) &= \partial^{e_i}\partial^\beta(fg)(x) = \partial^{e_i}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\beta-\gamma}g(x) \\
&= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^{e_i}\partial^\gamma f(x)\partial^{\beta-\gamma}g(x) + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{e_i}\partial^{\beta-\gamma}g(x) \\
&= \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^{\gamma+e_i}f(x)\partial^{\beta+e_i-(\gamma+e_i)}g(x) + \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\beta+e_i-\gamma}g(x) \\
&= \sum_{\gamma\le\beta+e_i,\gamma_i\ge1}\binom{\beta}{\gamma-e_i}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) + \sum_{\gamma\le\beta,\gamma_i\ge1}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&\quad + \sum_{\gamma\le\beta,\gamma_i=0}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1}\binom{\beta}{\gamma-e_i}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) + (\partial^\alpha f(x))g(x) \\
&\quad + \sum_{\gamma\le\beta,\gamma_i\ge1}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) + \sum_{\gamma\le\beta,\gamma_i=0}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1}\left(\binom{\beta}{\gamma-e_i}+\binom{\beta}{\gamma}\right)\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) + (\partial^\alpha f(x))g(x) \\
&\quad + \sum_{\gamma\le\beta,\gamma_i=0}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1}\left(\binom{\beta_i}{\gamma_i-1}+\binom{\beta_i}{\gamma_i}\right)\prod_{j\ne i}\binom{\beta_j}{\gamma_j}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&\quad + (\partial^\alpha f(x))g(x) + \sum_{\gamma\le\beta,\gamma_i=0}\binom{\beta}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&= \sum_{\gamma\le\beta,\gamma_i\ge1}\binom{\alpha_i}{\gamma_i}\prod_{j\ne i}\binom{\alpha_j}{\gamma_j}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&\quad + (\partial^\alpha f(x))g(x) + \sum_{\gamma\le\beta,\gamma_i=0}\binom{\alpha}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) \\
&= \sum_{\gamma\le\beta}\binom{\alpha}{\gamma}\partial^\gamma f(x)\partial^{\alpha-\gamma}g(x) + (\partial^\alpha f(x))g(x)
\end{aligned}$$

$$= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \partial^{\gamma} f(x) \partial^{\alpha - \gamma} g(x).$$

□