FOURIER ANALYSIS II. (spring 2020)

3. EXERCISES (Thursday 2.4)

NOTE: return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

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They should be returned at the latest during the day of the 'exercise class'. The course will be evaluated based on the returned exercises.

1. (a specially important exercise!) Prove in detail that $\rho(f_n, g) \to 0$ if and only if $p_N(f_n - g) \to 0$ for every $N \geq 0$.

2. (a specially important exercise!) Prove that a linear map $\lambda : \mathcal{S}(\mathbf{R}^d) \to \mathbf{C}$ is continuous if and only if there is an index $N \geq 0$ and constant $C < \infty$ such that

$$|\lambda(g)| \leq C p_N(g) \qquad \text{for all} \quad g \in \mathcal{S}(\mathbf{R}^d).$$

3. Assume that $f \in C^\infty(\mathbf{R}^d)$ satisfies for any multi-index α: there exists $M = M_\alpha$ ja $C = C_\alpha$ so that

$$|\partial^\alpha f(x)| \leq C(1 + |x|^2)^M \quad \text{for all} \;\; x \in \mathbf{R}^d.$$

Show that $fg \in \mathcal{S}(\mathbf{R}^d)$ for all $g \in \mathcal{S}(\mathbf{R}^d)$ and that the map $g \mapsto fg$ is a continuous linear map from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}(\mathbf{R}^d)$.

4. Show that the metric space $(\mathcal{S}(\mathbf{R}^d), \rho)$ (i.e. the Schwartz space of test functions equipped with the metric ρ) is complete.

5. (i) Let $a = (a_1, a_2) \in \mathbf{R}^2$ and $r > 0$. Show that $T \in \mathcal{S}'(\mathbf{R}^2)$, where

$$\langle T, g \rangle := \int_0^{2\pi} g(a + r(\cos(t), \sin(t))dt$$

when $g \in \mathcal{S}(\mathbf{R}^d)$.

(ii) Verify that $T \in \mathcal{S}'(\mathbf{R})$, where

$$\langle T, \phi \rangle := \sum_{k \in \mathbf{Z}} \phi(k^2).$$

6. Suppose the Fourier transform of a function $f \in L^1(\mathbf{R})$ satisfies the condition

$$|\widehat{f}(\xi)| \leq \frac{C}{(1 + |\xi|)^{1+a}}\,, \qquad \xi \in \mathbf{R},$$

for some constants $0 < a < 1$ and $C < \infty$. Show that then $f \in Lip_a(\mathbf{R})$, that is,

$$|f(x+h) - f(x)| \leq M\,|h|^a, \qquad x \in \mathbf{R}, \quad h \in \mathbf{R}.$$

[Hint: Recall the formula for the inverse Fourier transform, when $f,\ \widehat{f} \in L^1(\mathbf{R})$, and use it to estimate the continuity by dividing the integration to two parts: one over interval $[-A, A]$, one over the rest, and optimise A when h is given.]

7. Complete the proof of Riesz-Thorin interpolation theorem in case where $p = \infty$ (sketch the changes needed in the proof, not all details are needed).

8*[1] True or false: then there exists $\varphi \in \mathcal{S}(\mathbf{R})$ such that

$$\lim_{|x|\to\infty} e^{|x|}\varphi(x) = \infty \quad ?$$

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises