Algebra II. Exercise 9.
Solutions.

1. Consider the map $f\colon R[X] \to R$,

$$f(\sum_{i=0}^{n} a_i X^i) = a_0.$$

If $P, Q \in R[X]$, then the constant term of $P + Q$ is the sum of the constant terms of P and Q, and the constant term of $P \cdot Q$ is the product of the constant terms of P and Q. Also we have that $f(1) = 1$. Thus f is a ring homomorphism, and we may apply the homomorphism theorem (Exercise 2 last week). Since $\langle X \rangle$ consists of exactly those polynomials with constant term 0, that is, $\mathrm{Ker}(f) = \langle X \rangle$ and clearly $\mathrm{Im}(f) = R$, we immediately get that

$$R[X]/\langle X \rangle \cong R.$$

2. Let X be a linearly independent subset of the $\mathbb{Z}$-module $\mathbb{Q}$. Antithesis: there exist $\frac{a}{b}, \frac{c}{d} \in X$, $\frac{a}{b} \neq \frac{c}{d}$. Thus $b, d \neq 0$. Also $a \neq 0$ and $c \neq 0$, since 0 cannot belong to a linearly independent set (the linear combination $1 \cdot 0 = 0$, the coefficient $1 \neq 0$). Now

$$cb \cdot \frac{a}{b} - ad \cdot \frac{c}{d} = ac - ac = 0,$$

that is, a linear combination of the elements $\frac{a}{b}, \frac{c}{d}$ with coefficients in $\mathbb{Z}$ is 0, and the coefficients cb and ad are $\neq 0$. This is a contradiction with the assumption that X is a linearly independent set.
Thus a linearly independent set cannot contain more that one element.

If the $\mathbb{Z}$-module $\mathbb{Q}$ had a basis, then by the above, it would contain only one element, that is, $\mathbb{Q}$ would be generated by just one element, say $\frac{p}{q}$. Now

$$\langle \frac{p}{q} \rangle = \{n \cdot \frac{p}{q} \mid n \in \mathbb{Z}\}.$$

Since $\frac{p}{2q} \in \mathbb{Q}$, there should then exist $n \in \mathbb{Z}$, such that $\frac{p}{2q} = n \cdot \frac{p}{q}$, that is, $n = \frac{1}{2}$, which is a contradiction. Thus the $\mathbb{Z}$-module $\mathbb{Q}$ doesn't have a basis.

3. a) Suppose that M is an R/A-module. Define an action of the ring R in the group M by the formula

$$r.m = (r + A).m, \quad r \in R, \; m \in M.$$

We check that the conditions (M1)–(M4) are satisfied:

- $1.x = (1 + A).x = x$, since $1 + A$ is the unit element of the ring R/A.
- $(ab).x = (ab + A).x = ((a + A)(b + A)).x = (a + A).((b + A).x) = (a + A).(b.x) = a.(b.x)$.
- $(a+b).x = ((a+b)+A).x = ((a+A)+(b+A)).x = (a+A).x+(b+A).x = a.x + b.x$.
- $a.(x + y) = (a + A).(x + y) = (a + A).x + (a + A).y = a.x + a.y$.

Thus M is an R-module.

For example $\mathbb{Q}$ is a $\mathbb{Z}$-module (the action being the ordinary multiplication). If $\mathbb{Q}$ were a $\mathbb{Z}_n$-module, then

$$0 = [0].q = [n].q = ([1] + ... + [1]).q = [1].q + ... + [1].q = q + ... + q = nq$$

for every $q \in \mathbb{Q}$. Thus we would have $nq = 0$ for every $q \in \mathbb{Q}$, which is a contradiction. Thus $\mathbb{Q}$ is not a $\mathbb{Z}_n$-module.

b) If M is an R-module and $S \subset R$, we may define the action of S to be the restriction of the action of R. The conditions (M1)–(M4) clearly hold for this action of S, since they hold for the action of R.

For example $\mathbb{Z}$ is a $\mathbb{Z}$-module, the action being the ordinary multiplication. If $\mathbb{Z}$ were a $\mathbb{Q}$-module, then

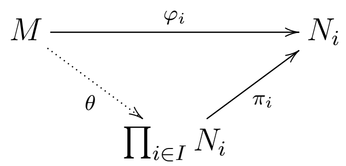
$$1 = 1.1 = (\frac{1}{2} + \frac{1}{2}).1 = \frac{1}{2}.1 + \frac{1}{2}.1,$$

where we should have $\frac{1}{2}.1 \in \mathbb{Z}$. This is a contradiction, since $x + x = 1$ doesn't hold for any integer x. Thus $\mathbb{Z}$ is not a $\mathbb{Q}$-module.

4. Define

$$\theta(m) = (\varphi_i(m))_{i \in I} \in \prod_i N_i.$$

The definition is ok, since the ith coordinate of the image element is $\varphi_i(m) \in N_i$. Furthermore the diagram



commutes.

We prove that the map is R-linear:

$$\theta(m_1 + m_2) = (\varphi_i(m_1 + m_2))_{i\in I} = (\varphi_i(m_1) + \varphi_i(m_2))_{i\in I}$$
$$= (\varphi_i(m_1))_{i\in I} + (\varphi_i(m_2))_{i\in I} = \theta(m_1) + \theta(m_2)$$

for every $m_1, m_2 \in M$. In the second step we used the linearity of the maps φ_i and in the third step we used the definition of the binary operation of the product module. Furthermore

$$\theta(a.m) = (\varphi_i(a.m))_{i\in I} = (a.\varphi_i(m))_{i\in I} = a.(\varphi_i(m))_{i\in I} = a.\theta(m)$$

for every $a \in R, m \in M$. In the second step we used the linearity of the maps φ_i and in the third step we used the definition of the action of R in the product module.

Uniqueness: because we want that $\pi_i(\theta(m)) = \varphi_i(m)$, then the ith coordinate of $\theta(m)$ must be $\varphi_i(m)$, and thus $\theta(m) = (\varphi_i(m))_{i\in I}$.

5. Let $b \in B$. Clearly $(f \circ g)(b) \in \mathrm{Im}(f)$. We also notice that $b - (f \circ g)(b) \in \mathrm{Ker}(g)$, because

$$g(b - (f \circ g)(b)) = g(b) - (g \circ f \circ g)(b) = g(b) - g(b) = 0.$$

In the second step we used the assumption $g \circ f = \mathrm{id}_A$.

We now prove that the map

$$\varphi\colon B \to \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$$
$$b \mapsto ((f \circ g)(b), b - (f \circ g)(b))$$

is an isomorphism. We first check that φ is R-linear; this follows directly from the assumption that f and g are R-linear (in which case also $f \circ g$ is), the definition of direct sum and the linearity of the action:

$$\varphi(b + b') = ((f \circ g)(b + b'), b + b' - (f \circ g)(b + b'))$$

$$= ((f \circ g)(b) + (f \circ g)(b'), b + b' - (f \circ g)(b) - (f \circ g)(b'))$$

$$= ((f \circ g)(b), b - (f \circ g)(b)) + ((f \circ g)(b'), b' - (f \circ g)(b')) = \varphi(b) + \varphi(b')$$

and

$$\varphi(r.b) = ((f \circ g)(r.b), r.b - (f \circ g)(r.b)) = (r.(f \circ g)(b), r.b - r.(f \circ g)(b))$$

$$= (r.(f \circ g)(b), r.(b - (f \circ g)(b))) = r.((f \circ g)(b), b - (f \circ g)(b)) = r.\varphi(b).$$

Next we prove that φ is injective: If

$$((f \circ g)(b), b - (f \circ g)(b)) = ((f \circ g)(b'), b' - (f \circ g)(b')),$$

then by looking at the first coordinates we get $(f \circ g)(b) = (f \circ g)(b')$, and then by looking at the second coordinates we get $b = b'$. Thus φ is injective. Finally we prove that φ is surjective: Let $(b_1, b_2) \in \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$. Thus $b_1 = f(a_1)$ for some $a_1 \in A$ and $g(b_2) = 0$. Choose $b = b_1 + b_2 \in B$. We notice that

$$(f \circ g)(b_1 + b_2) = (f \circ g)(b_1) + (f \circ g)(b_2) = (f \circ g \circ f)(a_1) + 0 = f(a_1) = b_1,$$

hence

$$\varphi(b) = ((f \circ g)(b_1 + b_2), b_1 + b_2 - (f \circ g)(b_1 + b_2)) = (b_1, b_1 + b_2 - b_1) = (b_1, b_2).$$

This proves surjectivity.

6. Choose $R = \mathbb{R}$, $M = \mathbb{R}$ and $N_i = \mathbb{R}$ for every $i \in \mathbb{N}$, and $\varphi_i = \mathrm{id}: M \to N_i$ for every $i \in \mathbb{N}$. Denote the projections by $\pi_i: \bigoplus_{i \in \mathbb{N}} N_i \to N_i$, $i \in \mathbb{N}$.
We try to define a function $\theta: \mathbb{R} \to \bigoplus_{i \in \mathbb{N}} \mathbb{R}$, such that $\pi_i \circ \theta = \mathrm{id}: \mathbb{R} \to \mathbb{R}$. Consider for example the element $1 \in \mathbb{R}$. We notice that from the condition $\pi_i \circ \theta = \mathrm{id}$ it follows that we should have $\theta(1) = (1, 1, 1, ...)$. This element belongs to the direct product, but not to the direct sum (in the direct sum only finitely many coordinates can be $\neq 0$). Thus such a function θ doesn't exist.