Algebra II

Preliminary exercises 10

1. Let R be a commutative ring. Verify that the multiplication is a bilinear map $R \times R \to R$.

2. a) The tensor product of two R-modules M, N is defined as the quotient module C/D (the notation is explained in the lecture material). By investigating the elements, which generate the submodule D, verify that the symbols $x \otimes y$ satisfy the following identities: $(x+y) \otimes z = (x \otimes z) + (y \otimes z)$, $x \otimes (z+w) = (x \otimes z) + (x \otimes w)$, $(a.x) \otimes z = a.(x \otimes z)$ and $x \otimes (a.z) = a.(x \otimes z)$, where $x, y \in M,\ z, w \in N,\ a \in R$.

b) Study the example in the lecture material, which proves that $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \{0\}$, if g.c.d.$(m, n) = 1$. Explain the steps in the proof, using for example the identities of item a) above.

3. Preparation for exercise 5 (of the "actual" exercises). Suppose that M, M', N are R-modules and $\varphi \colon M \to M'$ is an R-linear map. Prove that the composite map

$$f = \eta' \circ (\varphi \times \mathrm{id}) \colon M \times P \to M' \times P \to M' \otimes P$$

$$(m, p) \mapsto (\varphi(m), p) \mapsto \varphi(m) \otimes p$$

is R-bilinear.

4. a) Consider the following elements of the module $\mathbb{R}^2 \otimes \mathbb{R}^{3\times 2}$:

$$a = (-1, 2) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = (0, 1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and } c = (-1, 2) \otimes \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$$

Calculate (simplify) $a + b$, $3a$ and $a - 2c$.

b) Based on the course material, we know that the elements of the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^{3\times 2}$ can be written in the form $\sum_i x_i \otimes y_i$, where $x_i \in \mathbb{R}^2$ and $y_i \in \mathbb{R}^{3\times 2}$ for all i. Show that this presentation is not unique (that is, elements can be written in the aforementioned form in several different ways).