Algebra II

Exercise 10 (2.4.2020)

1. Prove that every Abelian group G is isomorphic to a factor group of some free Abelian group. [Hint: Choose some set of generators X for the group G, consider the free module $\mathbb{Z}^{(X)}$ and apply the homomorphism theorem to a suitable linear map $\varphi\colon \mathbb{Z}^{(X)} \to G$.]

2. Verify that the dot product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x_1, ..., x_n) \cdot (y_1, ..., y_n) = x_1 y_1 + ... + x_n y_n$, is an $\mathbb{R}$-bilinear map.

3. Prove that the canonical map

$$\eta\colon M \times N \to M \otimes_R N$$

associated with the tensor product is R-bilinear.

4. Suppose that M is a finite Abelian group. Prove that

$$\mathbb{Q} \otimes_{\mathbb{Z}} M = \{0\}.$$

5. Suppose that M, M' and N are R-modules and suppose also that M and M' are isomorphic as R-modules. Prove that

$$M \otimes N \cong M' \otimes N.$$

6. Consider the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ of $\mathbb{Z}$-modules.

a) Prove that there exists a $\mathbb{Z}$-linear map $\varphi\colon \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$, for which $\varphi(x \otimes y) = xy$.

b) Prove that the map $\psi\colon \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\psi(x) = x \otimes 1$, is surjective and it is the inverse map of φ (and thus $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$).