Fourier Analysis II

Spring 2020

Homework 2

Exercise session: Thu 26 March; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1-2. (a specially important exercise!) Let $\alpha \in \mathbb{N}_0^d$ be a multi-index. Prove with all details that if $f \in \mathcal{S}(\mathbb{R}^d)$, then

(i) $x^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$ and $\partial^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d)$.

(ii) $\widehat{f} \in C^\infty(\mathbb{R}^d)$.

(iii) $(\partial^\alpha f)\widehat{\ }(\xi) = (i\,\xi)^\alpha \widehat{f}(\xi)$ (note that one defines $i^\alpha := i^{|\alpha|}$).

(iv) Apply part (iii) by choosing suitable multi-indices α to verify that $\widehat{f}$ decays any polynomial rate, i.e. for any $N \geq 1$ there is a constant C so that $|\widehat{f}(\xi)| \leq C(1+|\xi|^2)^{-N}$.

Proof. For ease of notation, let's say that a function f satisfies the $(*)$**-condition** if

$$\sup_{x\in\mathbb{R}^d} (1+|x|)^N |f(x)| < \infty \quad \text{for all} \quad N \geq 0.$$

Thus a function is in $\mathcal{S}(\mathbb{R}^d)$ if it and all its derivatives satisfy the $(*)$-condition.
(i) If $\alpha \in \mathbb{N}_0^d$ is a multi-index, we recall that

$$x^\alpha = x_1^{\alpha_1} x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

Also, notice that if $\alpha, \beta \in \mathbb{N}_0^d$ are multi-indices and $x = (x_1, \dots, x_d)$, then

$$\partial^\alpha x^\beta = \begin{cases} \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha}, & \text{if } \alpha \leq \beta \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Using (1) and the general Leibniz formula for differentiation of products from Exercise 8 in the previous set, we have that

$$\begin{aligned} \sup_{|\beta|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|)^N |\partial^\beta (x^\alpha f)(x)| &= \sup_{|\beta|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|)^N \left| \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \partial^\gamma x^\alpha \partial^{\beta-\gamma} f(x) \right| \\ &= \sup_{|\beta|\leq N} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \sup_{x\in\mathbb{R}^d} |(1+|x|)^N \partial^\gamma x^\alpha \partial^{\beta-\gamma} f(x)| \\ &\leq C_{\alpha,\beta,\gamma,N} < \infty, \end{aligned}$$

for all $N \geq 0$, $\beta \in \mathbb{N}_0^d$.
It is also easy to see that $\partial^\alpha f$ is in $\mathcal{S}(\mathbb{R}^d)$, since $\partial^\beta \partial^\alpha f = \partial^{\beta+\alpha} f$, which satisfies $(*)$-condition for every $N \geq 0$, $\beta \in \mathbb{N}_0^d$.
(ii) We apply Theorem 9.4 of the lecture notes to show that $\partial^\alpha \widehat{f}(\xi) = ((-ix)^\alpha f(x))\widehat{\ }(\xi)$. We assume that the formula holds for some multi-index α. Let $e_j = (0, \ldots, 1, \ldots, 0)$, where jth index is 1. According to part **(i)** $g(x) = (-ix_j)(-ix)^\alpha f(x) \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and therefore

$$\begin{aligned}\partial^{\alpha+e_j} \widehat{f}(\xi) &= \frac{\partial}{\partial \xi_j}(\partial^\alpha \widehat{f}(\xi)) = \frac{\partial}{\partial \xi_j}((-ix)^\alpha f(x))\widehat{\ }(\xi) \\ &= (-ix_j(-ix)^\alpha f(x))\widehat{\ }(\xi) = ((-ix)^{\alpha+e_j} f(x))\widehat{\ }(\xi).\end{aligned}$$

The formula therefore holds for $\alpha+e_j$, and by induction, for any multi-index. In particular, $\widehat{f} \in C^\infty(\mathbb{R}^d)$.
(iii) By induction it is enough to prove that

$$(\partial_j f)\widehat{\ }(\xi) = (i\xi_j)\widehat{f}(\xi)$$

for all j. To do this we use integration by parts to obtain that

$$\int_{\mathbb{R}^d} (\partial_j f)(x) e^{-i\xi x} dx = -\int_{\mathbb{R}^d} f(x) \partial_j e^{-i\xi x} dx = i\xi_j \int_{\mathbb{R}^d} f(x) e^{-i\xi x} dx,$$

which is what we wanted.
(iv) Let $N \geq 1$ be fixed. For any fixed multi-index α we know by part **(i)** that $\partial^\alpha f \in \mathcal{S}(\mathbb{R}^d)$. Since the Fourier transform maps any $L^1(\mathbb{R}^d)$ function into $L^\infty(\mathbb{R}^d)$, we know by part **(iii)** that $(i\xi)^\alpha \widehat{f}(\xi)$ is in $L^\infty(\mathbb{R}^d)$ for every $\alpha \in \mathbb{N}_0^d$. Hence we get the bounds

$$|\xi^\alpha||\widehat{f}(\xi)| \leq C_\alpha.$$

As $(1+|\xi|^2)^N$ is a polynomial, we can express it as a finite sum of terms of form ξ^α. Hence there exists a constant C_N such that

$$(1+|\xi|^2)^N |\widehat{f}(\xi)| \leq C_N.$$

Dividing by $(1+|\xi|^2)^N$ gives $|\widehat{f}(\xi)| \leq C_N(1+|\xi|^2)^{-N}$ as wanted. □

3. (a specially important exercise!) Apply the previous exercise and verify carefully that

$$\text{if} \quad f \in \mathcal{S}(\mathbb{R}^d), \quad \text{then} \quad \widehat{f} \in \mathcal{S}(\mathbb{R}^d).$$

Proof. We proved in the previous exercise that $\widehat{f} \in C^\infty(\mathbb{R}^d)$. It remains to show that all the derivatives satisfy the $(*)$-condition.
The previous exercise already implies that for any Schwartz function g the function $\widehat{g}$ already satisfies the $(*)$-condition. We also showed that for any multi-index α

$$\partial^\alpha \widehat{f}(\xi) = ((-ix)^\alpha f(x))\widehat{}(\xi)$$

and $(-ix)^\alpha f(x)$ is a Schwartz function. We see that any derivatives of $\widehat{f}$ is Fourier transform of a Schwartz function and therefore satisfies the $(*)$-condition. □

4. Which of the following functions belong to $\mathcal{S}(\mathbb{R}^d)$?

(i) $f(x) = (1 + |x|^2)^{-1}$. **(ii)** $f(x) = e^{-|x|^2}$.

(iii) $f(x) = e^{-|x|^2} \cos(e^{|x|^2})$.

Proof. **(i)** This function does not belong in $\mathcal{S}(\mathbb{R}^d)$. We see that

$$\sup_{x\in\mathbb{R}^d} (1 + |x|^2)^2 |f(x)| = \sup_{x\in\mathbb{R}^d} 1 + |x|^2 = \infty.$$

(ii) This function belongs in $\mathcal{S}(\mathbb{R}^d)$.
This function is smooth as it is a composition of smooth functions. Note that $f(x)$ itself satisfies $(*)$-condition, since the exponential functions grows faster than any polynomial. Now

$$\partial_j f(x) = -2x_j e^{-|x|^2}.$$

By induction we see that $\partial^\alpha f(x)$ is some polynomial times $f(x)$ for every multi-index α. But these types of functions also satisfy $(*)$-condition, since we proved that if f satisfies $(*)$-condition then $x_j f$ also satisfies $(*)$-condition and we can use induction to prove this for any polynomial in place of x_j. Thus all the partial derivatives satisfy $(*)$-condition and $f \in \mathcal{S}(\mathbb{R}^d)$.
(iii) This function does not belong in $\mathcal{S}(\mathbb{R}^d)$. Considering its partial derivatives shows that

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} (1 + |x|^2)|\partial_1 f(x)| &\geq \sup_{x\in\mathbb{R}^d} |-2x_1 e^{-|x|^2} \cos(e^{|x|^2}) - e^{-|x|^2} \sin(e^{|x|^2}) e^{|x|^2} 2x_1| \\ &= \sup_{x\in\mathbb{R}^d} |-2x_1 e^{|x|^2} \cos(e^{|x|^2}) - 2x_1 \sin(e^{|x|^2})| = \infty,\end{aligned}$$

because the term $2x_1 \sin(e^{|x|^2})$ is not bounded as $x_1 \to \infty$. □

5. Compute the integral $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$ by first computing the Fourier transform of the characteristic function $\chi_{[-1,1]}$.

Proof. Recall from Exercise 1 in the previous set that

$$\widehat{\chi}_{[-1,1]}(\xi) = \begin{cases} \frac{2\sin(\xi)}{\xi}, & \text{if } \xi \neq 0 \\ 2, & \text{if } \xi = 0. \end{cases}$$

By Theorem 10.1 of the lecture notes we also know that for any $f \in L^2(\mathbb{R})$

$$2\pi \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi.$$

Now we use these facts to see that

$$4\pi = 2\pi \int_{-1}^{1} 1^2 dx = \int_{-\infty}^{\infty} \left(\frac{2\sin\xi}{\xi}\right)^2 d\xi$$

and we can solve that

$$\int_{-\infty}^{\infty} \left(\frac{\sin\xi}{\xi}\right)^2 d\xi = \pi.$$

□

6. Assume that $f \in \mathcal{S}(\mathbb{R}^d)$. **(i)** Compute the Fourier transform of the Laplacian $\Delta f := \left(\sum_{j=1}^{d} \left(\frac{\partial}{\partial x_j}\right)^2\right) f$ in terms of $\widehat{f}$.

(ii) Show that $\dfrac{f(x)}{1+|x|^2} \in \mathcal{S}(\mathbb{R}^d)$.

Proof. **(i)** Using part **(iii)** from Exercise 1-2, we find that

$$\widehat{\Delta f}(\xi) = \sum_{j=1}^{d} \widehat{\partial_{x_j}^2 f}(\xi) = \sum_{j=1}^{d} (i\xi_j)^2 \widehat{f}(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

(ii) Let us say that a function $R(x)$ is a **good rational function** if

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ does not take the value zero. Especially $(1+|x|^2)^{-1}$ is a good rational function. The claim of the exercise now follows from the following two results:

Claim 1. *If $R(x)$ is a good rational function and f satisfies $(*)$-condition from Exercise 1-2, then $R(x)f(x)$ satisfies $(*)$-condition.*

Proof. If $R(x)$ is a good rational function, then $|R(x)| \leq C(1+|x|)^M$ for some constants $C, M \geq 0$. Thus

$$\sup_{x\in\mathbb{R}^d}(1+|x|)^N|R(x)f(x)| \leq \sup_{x\in\mathbb{R}^d} C(1+|x|)^{N+M}|f(x)| < \infty$$

for all $N \geq 0$. This proves the Claim 1. □

Claim 2. *If a function is of the form $R(x)f(x)$ with R a good rational function and $f \in \mathcal{S}$, then all of its first-order derivatives are also sums of functions of the same form.*

Proof. We simply compute that

$$\partial_{x_j} R(x)f(x) = \frac{(\partial_{x_j}P(x))Q(x) - P(x)\partial_{x_j}Q(x)}{Q(x)^2}f(x) + R(x)\partial_{x_j}f(x),$$

which is the desired form. □

The claim of the exercise now follows by induction. By Claim 1, the function $(1+|x|^2)^{-1}f(x)$ satisfies $(*)$-condition. By Claim 2 and 1, so do its first order derivatives. Continuing this argument we find that all the derivatives satisfy $(*)$-condition, so $(1+|x|^2)^{-1}f(x) \in \mathcal{S}$. □

7. Use Fourier transform to find a solution formula for the partial differential equation

$$\Delta f - f = g$$

for given $g \in \mathcal{S}(\mathbb{R}^d)$ and show that also the solution f lies in $\mathcal{S}(\mathbb{R}^d)$.

Proof. If $f \in \mathcal{S}$ satisfies the equation

$$\Delta f - f = g, \quad \Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_d}\right)^2,$$

we can take the Fourier transform of both sides to find that

$$-(|\xi|^2+1)\widehat{f}(\xi) = \widehat{g}(\xi).$$

Recall from Exercise 3 that we know that $\widehat{f}(\xi), \widehat{g}(\xi) \in \mathcal{S}$. We can now solve the Fourier transform of f:

$$\widehat{f}(\xi) = -(1 + |\xi|^2)^{-1}\widehat{g}(\xi). \tag{2}$$

By previous exercise, we know that $-(1 + |\xi|^2)^{-1}\widehat{g}(\xi) \in \mathcal{S}$. Thus it is possible to take the inverse Fourier transform $\mathcal{F}^{-1}$ to find the solution f:

$$f(x) = \mathcal{F}^{-1}[-(1 + |\xi|^2)^{-1}\widehat{g}(\xi)](x).$$

This solves the original equation so we are done. □

8. **(i)** Specialize in the previous exercise to dimension $d = 1$ and show that the solution is given by the convolution

$$f(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|}g(y)dy.$$

(ii) Given $\varepsilon > 0$, show that one may pick $g \in \mathcal{S}(\mathbb{R})$ so that the solution f satisfies $\|f\|_{L^2(\mathbb{R})} < \varepsilon\|g\|_{L^2(\mathbb{R})}$.

Proof. **(i)** We define function h as $h(x) = e^{-|x|}$. We saw in Exercise 3 of the previous set that $\widehat{h}(\xi) = \frac{2}{1+\xi^2}$. Therefore

$$\widehat{f}(\xi) = -\frac{1}{2}\widehat{g}(\xi)\widehat{h}(\xi).$$

As the Fourier transform of the convolution is the product of their Fourier transforms, we have

$$f(x) = -\frac{1}{2}(g * h)(x) = -\frac{1}{2}\int_{-\infty}^{\infty} g(y)e^{-|x-y|}dy.$$

(ii) Let M be a constant. Let h be a $C_c^\infty(\mathbb{R})$ function with $h(x) = 0$ for $|x| \leq M$ that is not identically 0. As compactly supported functions are Schwartz functions, we know that there exists a Schwartz function g with $\widehat{g} = h$ (see Theorem 9.14 of the lecture notes). Now using (2) for $d = 1$ we can estimate

$$\begin{aligned}
\|f\|_{L^2(\mathbb{R})} &= (2\pi)^{-1/2}\|\widehat{f}\|_{L^2(\mathbb{R})} = (2\pi)^{-1/2}\left(\int_{-\infty}^{\infty}(1 + |\xi|^2)^{-2}|\widehat{g}(\xi)|^2 d\xi\right)^{1/2} \\
&\leq (2\pi)^{-1/2}\left(\int_{-\infty}^{\infty}(1 + M^2)^{-2}|\widehat{g}(\xi)|^2 d\xi\right)^{1/2} = (1 + M^2)^{-1}(2\pi)^{-1/2}\|\widehat{g}\|_{L^2(\mathbb{R})} \\
&= (1 + M^2)^{-1}\|g\|_{L^2(\mathbb{R})}.
\end{aligned}$$

As $\|g\|_{L^2(\mathbb{R})} \neq 0$ and M was arbitrary, the claim follows.

□