Exercises 4

Please write your name.

Here k is an algebraically closed field.

Exercise 1 (25 points)**.** Assume that characteristic of k is zero. Compute the singular locus of the following algebraic sets

- $Z_1 := V(x^2 - x^4 - y^4) \subset \mathbb{A}^2_k$,
- $Z_2 := V(xy - x^6 - x^6) \subset \mathbb{A}^2_k$,
- $Z_3 := V(x^3 - y^2 - x^4 - y^4) \subset \mathbb{A}^2_k$,
- $Z_4 := V(x^2y + xy^2 - x^4 - y^4) \subset \mathbb{A}^2_k$,
- $Z_5 := V(xy^2 - z^2) \subset \mathbb{A}^3_k$,
- $Z_6 := V(x^2 + y^2 - z^2) \subset \mathbb{A}^3_k$,
- $Z_7 := V(xy + x^3 + y^3) \subset \mathbb{A}^3_k$,
- $Z_8 := V(X_0^d + X_1^d + \ldots + X_n^d) \subset \mathbb{P}^n_k$,
- $Z_9 := V(X_0^2 - X_1^2) \subset \mathbb{P}^2_k$ (note the dimension of the projective space).

Exercise 2 (10 points)**.** Let $Z = V(f) \subset \mathbb{A}^n_k$ be an hypersurface. Show that if Z is connected (i.e if $Z = U \cup V$ with U and V non-empty open subsets of Z, then $U \cap V \neq 0$) and reducible, then the singular locus Z^{sing} is non-empty.

Exercise 3 (15 points)**.** Assume that characteristic of k is zero. Let $V(F) \subset \mathbb{P}^n_k$ be a projective hypersurface. Show that $p \in Z$ is a singular point if and only if the partial derivatives with respect to the homogeneous coordinate X_i vanishes at p for any i, i.e. $\frac{\partial F}{\partial X_i}(p) = 0$ for any i.

(Hint: (1) pass to an affine cover containing p and use the embedded tangent space, (2) use the fact that for any homogeneous polynomial $F \in k[X_0, \ldots, X_n]$ of degree d

$$\sum_{i=0}^{n} X_i \frac{\partial F}{\partial X_i} = d \cdot F$$

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