Exercises 3

Here k is an algebraically closed field.

1 Solutions

Exercise 1 (15 points)**.** Take the 2-dimensional projective space $\mathbb{P}^2_k$ with coordinate ring $k[X,Y,Z]$. Consider the following projective plane curves

(a) $W_1 := V(Y^2Z - 2X^3 + 3XZ^2 + Z^3) \subset \mathbb{P}^2_k$.

(b) $W_2 := V(XZ^3 - (X^2+Z^2)Y^2) \subset \mathbb{P}^2_k$.

Write down their restrictions to the 3 affine pieces $\mathbb{P}^2_X$, $\mathbb{P}^2_Y$, $\mathbb{P}^2_Z$ and determine the intersection of the curve with the hyperplanes $V(X)$, $V(Y)$, $V(Z)$.

Exercise 2 (10 points)**.** Show that any regular morphism $\varphi : \mathbb{P}^1_k \to \mathbb{A}^1_k$ is constant (Hint: restrict the morphism to the affine pieces $\mathbb{P}^1_{X_0}$ and $\mathbb{P}^1_{X_1}$, show that $\varphi_{\mathbb{P}^1_{X_i}}$ must be a polynomial, see Corollary 7 in Week 2. and then ....).

Exercise 3 (10 points)**.** Set $\mathbb{N}^3_d := \{(a,b,c) \in \mathbb{Z} \,|\, a+b+c = d \text{ and } a,b,c \geq 0\}$. Then we can define the Veronese morphism

$$\nu_{d,2} : \mathbb{P}^2_k \to \mathbb{P}\left(\mathrm{Sym}^d(k^3)\right) \cong \mathbb{P}_k^{\binom{d+2}{2}-1}$$

as it follows

$$[X_0, X_1, X_2] \mapsto \left[\ldots, \underset{(d_0,d_1,d_2)\in\mathbb{N}^3_d}{X_0^{d_0}X_1^{d_1}X_2^{d_2}}, \ldots\right]$$

We remark that the definition is equivalent to the one given in Notes 3. It is a closed embedding, i.e. its image is closed and it induces an isomorphism between the source and its own image: $\mathbb{P}^1_k \cong \mathrm{Im}(\nu_{d,2})$. Denote by $k[\ldots, Y_{(d_0,d_1,d_2)}, \ldots]$ the polynomial ring of $\mathbb{P}_k^{\binom{d+2}{2}-1}$.

(a) Fix an hyperplane

$$\mathbb{P}_k^{\binom{d+2}{2}-1} \supset H = V\left(\sum_{(d_0,d_1,d_2)\in\mathbb{N}^3_d} a_{(d_0,d_1,d_2)} Y_{(d_0,d_1,d_2)}\right), \quad \text{where } a_{(d_0,d_1,d_2)} \in k$$

in $\mathbb{P}^d_k$. Show that $\nu_{d,2}^{-1}(H)$ is a plane curve of degree d, i.e. $\nu_d^{-1}(H) = V(F)$ with $F \in k[X_0,X_1,X_2]$ homogeneous polynomial of degree d. (Suggestion: give a look to the case $d = 2$, where you can write the Veronese morphism more easily.)

(b) Show that for any $F \in k[X_0,X_1,X_2]$ homogeneous polynomial of degree $d > 0$ there exists an hyperplane $H \subset \mathbb{P}_k^{\binom{d+2}{2}-1}$ such that $\nu_{d,2}^{-1}(H) = V(F)$. (Suggestion: give a look to the case $d = 2$, where you can write the Veronese morphism more easily.)

(c) Deduce that if $F \in k[X_0, X_1, X_2]$ homogeneous polynomial of degree $d > 0$, then $\mathbb{P}^2_k \setminus V(F)$ is an affine variety, i.e. isomorphic to an irreducible affine algebraic set.

Exercise 4 (15 points)**.** Consider the following quadric $Z := V(X_0X_3 - X_1X_2) \subset \mathbb{P}^3_k$.

(a) Find two disjoint lines in Z.

(b) Deduce that Z and $\mathbb{P}^2_k$ are not isomorphic. (Hint: use exercises 2 and 3(c))

(c) Show that Z and $\mathbb{P}^2_k$ are birational.