Fourier Analysis II

Spring 2020

Homework 3

Exercise session: Thu 2 April; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. (a specially important exercise!) Prove in detail that $\rho(f_n, g) \to 0$ if and only if $p_N(f_n - g) \to 0$ for every $N \geq 0$.

Proof. Direction $\Rightarrow$. We assume that $\rho(f_n, g) \to 0$. Suppose to the contrary that there exists N_0 such that $p_{N_0}(f_n - f) \geq \epsilon$ for infinitely many n. Then for such n

$$\rho(f_n, g) = \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \geq 2^{-N_0} \frac{p_{N_0}(f_n - f)}{1 + p_{N_0}(f_n - f)} \geq 2^{-N_0} \frac{\epsilon}{1 + \epsilon} > 0,$$

a contradiction.
Direction $\Leftarrow$. Assume that $p_N(f_n - f) \to 0$ for all $N \geq 0$. Let $\epsilon > 0$ and N_0 be a large number to be chosen later. We estimate that

$$\begin{aligned}
\rho(f_n, f) &= \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \\
&= \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + \sum_{N=N_0+1}^{\infty} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} \\
&\leq \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + \sum_{N=N_0+1}^{\infty} 2^{-N} \\
&= \sum_{N=0}^{N_0} 2^{-N} \frac{p_N(f_n - f)}{1 + p_N(f_n - f)} + 2^{-N_0}.
\end{aligned}$$

If we choose N_0 large enough, the second term will be less than $\epsilon/2$. The first term can then be estimated since it contains only finitely many terms, and for each term we can use our assumption to choose $n \geq n_0$ large enough so that $p_N(f_n - f)$ is as small as we wish. Thus we can also bound the first term by $\epsilon/2$ if we want to, which proves the claim. □

2. (a specially important exercise!) Prove that a linear map $\lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is continuous if and only if there is an index $N \geq 0$ and constant $C < \infty$ such that

$$|\lambda(g)| \leq C p_N(g) \qquad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^d).$$

Proof. By translation invariance of λ and the metric $\rho(f,g) = \rho(f-g,0)$, the fact that λ is continuous is equivalent with the fact that is continuous at zero. With this in mind, we proceed.
Direction $\Leftarrow$. Let f_n be a sequence converging to zero in the metric ρ. By Exercise 1 we know that $p_N(f_n) \to 0$ as $N \to \infty$. Since $|\lambda(f_n)| \leq C p_N(f_n)$, we also know that $\lambda(f_n) \to 0$ so λ is continuous at zero.
Direction $\Rightarrow$. We make a proof by contradiction. Suppose that for every $N \geq 0$ and constant $C = N$ there is a sequence f_N of functions such that

$$|\lambda(f_N)| \geq N p_N(f_N).$$

By linearity of λ, we can scale this to assume that $p_N(f_N) = 1/N$. We prove that then $f_N \to 0$ in the metric ρ. By Exercise 1 it is enough to prove that $p_M(f_N) \to 0$ for every M. But this follows from the fact that if $N \geq M$, then since the p_M are increasing in M we get

$$p_M(f_N) \leq p_N(f_N) = 1/N \to 0.$$

However,

$$|\lambda(f_N)| \geq 1,$$

and this is a contradiction since we assumed that λ is continuous at zero. □

3. Assume that $f \in C^\infty(\mathbb{R}^d)$ satisfies for any multi-index α: there exists $M = M_\alpha$ and $C = C_\alpha$ so that

$$|\partial^\alpha f(x)| \leq C(1+|x|^2)^M \quad \text{for all } \ x \in \mathbb{R}^d. \tag{1}$$

Show that $fg \in \mathcal{S}(\mathbb{R}^d)$ for all $g \in \mathcal{S}(\mathbb{R}^d)$ and that the map $g \mapsto fg$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$.

Proof. The function fg is smooth as a product of smooth functions, and linearity is trivial. We need to estimate the norms $p_N(fg)$. To prove that $fg \in \mathcal{S}(\mathbb{R}^d)$, we need to show that the norms are finite, and for continuity we need to estimate them by the norms of g. Using (1) and the general Leibniz formula from Exercise 8 in the exercise set 1 we get:

$$\begin{aligned}
p_N(fg) &= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |\partial^\alpha (fg)(x)| \\
&= \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N \left| \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \partial^{\alpha-\beta} g(x) \right| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^d} (1+|x|^2)^N |\partial^\beta f(x) \partial^{\alpha-\beta} g(x)|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sup_{x\in\mathbb{R}^d} C(1+|x|^2)^{N+M}|\partial^{\alpha-\beta}g(x)| \\
&\leq \sup_{|\alpha|\leq N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} C p_{N+M}(g) \\
&= C2^N p_{N+M}(g).
\end{aligned}
$$

Here the constants C and M are defined by

$$
C = \max\{C_\alpha : |\alpha| \leq N\} \quad \text{and} \quad M = \max\{M_\alpha : |\alpha| \leq N\}.
$$

Now the map $g \mapsto fg$ is continuous by Theorem 12.2 in the lecture notes. □

4. Show that the metric space $(\mathcal{S}(\mathbb{R}^d), \rho)$ (i.e. the Schwartz space of test functions equipped with the metric ρ) is complete.

Proof. Suppose that we have a Cauchy sequence f_n in the metric ρ. Then f_n will also be Cauchy in each of the norms p_N, which follows from the fact that

$$
\frac{p_N(f_n - f_m)}{1 + p_N(f_n - f_m)} \leq 2^N \sum_{N=0}^{\infty} 2^{-N} \frac{p_N(f_n - f_m)}{1 + p_N(f_n - f_m)} = 2^N \rho(f_n, f_m),
$$

so if $\rho(f_n, f_m)$ is small then $p_N(f_n - f_m)$ must be small as well. Choosing $N = 0$, we find that f_n is Cauchy in the sup-norm and thus has a limit f in the sup-norm. For each multi-index α, the sequence $\partial^\alpha f_n$ is also Cauchy in the sup-norm and thus converges to some function g_α. By basic results about uniformly converging sequences of functions we know that $\partial^\alpha f = g_\alpha$. We must still prove that f_n converges to f in the metric ρ. Let $\epsilon > 0$ be arbitrary. Then there is $n_0 \in \mathbb{N}$ such that $p_N(f_m - f_n) \leq \epsilon$ when $m, n \geq n_0$. Thus

$$
(1 + |x|^2)^N |\partial^\alpha f_n(x) - \partial^\alpha f_m(x)| \leq \epsilon.
$$

By uniform convergence, we let $n \to \infty$ to get that

$$
(1 + |x|^2)^N |\partial^\alpha f(x) - \partial^\alpha f_m(x)| \leq \epsilon.
$$

This shows that $p_N(f - f_m) \to 0$ as $m \to \infty$ for every $N \geq 0$. Hence, by Exercise 1 we can conclude that $\rho(f_n, f) \to 0$ as $n \to \infty$. Obviously also f is in the Schwartz space by $p_N(f) \leq p_N(f - f_m) + p_N(f_m) < \infty$ for every $N \geq 0$. □

5. **(i)** Let $a = (a_1, a_2) \in \mathbb{R}^2$ and $r > 0$. Show that $T \in \mathcal{S}'(\mathbb{R}^2)$, where

$$\langle T, g \rangle := \int_0^{2\pi} g(a + r(\cos(t), \sin(t))dt$$

when $g \in \mathcal{S}(\mathbb{R}^2)$.

(ii) Verify that $T \in \mathcal{S}'(\mathbb{R})$, where

$$\langle T, \phi \rangle := \sum_{k \in \mathbb{Z}} \phi(k^2)$$

when $\phi \in \mathcal{S}(\mathbb{R})$.

Proof. **(i)** It is clear that T is linear. As $|g(x)| \leq p_0(g)$, we can estimate

$$\begin{aligned} |\langle T, g \rangle| &= \left| \int_0^{2\pi} g(a + r(\cos(t), \sin(t))dt \right| \\ &\leq \int_0^{2\pi} |g(a + r(\cos(t), \sin(t))|dt \\ &\leq \int_0^{2\pi} p_0(g)dt \\ &= 2\pi p_0(g). \end{aligned}$$

This proves by Exercise 2 that $T \in \mathcal{S}'(\mathbb{R}^2)$.
(ii) The linearity is clear again. We know that $|\phi(x)| \leq p_1(\phi)/(1 + |x|^2)$. We may now estimate

$$|\langle T, \phi \rangle| = \left| \sum_{k \in \mathbb{Z}} \phi(k^2) \right| \leq \sum_{k \in \mathbb{Z}} |\phi(k^2)| \leq \sum_{k \in \mathbb{Z}} \frac{p_1(\phi)}{1 + k^4} = Cp_1(\phi).$$

Here $C = \sum_{k \in \mathbb{Z}} (1 + k^4)^{-1} < \infty$ is the value of the sum. Exercise 2 implies that $T \in \mathcal{S}'(\mathbb{R})$. □

6. Suppose the Fourier transform of a function $f \in L^1(\mathbb{R})$ satisfies the condition

$$|\widehat{f}(\xi)| \leq \frac{C}{(1 + |\xi|)^{1+a}}, \qquad \xi \in \mathbb{R},$$

for some constants $0 < a < 1$ and $C < \infty$. Show that then $f \in Lip_a(\mathbb{R})$, that is,

$$|f(x + h) - f(x)| \leq M\, |h|^a, \qquad x \in \mathbb{R}, \quad h \in \mathbb{R}.$$

Proof. Fix $x, h \in \mathbb{R}$ and let $y = x + h$. In the following the notation $a \lesssim b$ means that $a \leq cb$ holds for $c > 0$. By the Fourier inversion theorem we begin, and then continue by using the assumptions to achieve that

$$\begin{aligned}
|f(x) - f(y)| &= \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{i\xi x}\widehat{f}(\xi)d\xi - \int_{\mathbb{R}} e^{i\xi y}\widehat{f}(\xi)d\xi\right) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} (e^{i\xi x} - e^{i\xi y})\widehat{f}(\xi)d\xi \\
&\lesssim \int_{\mathbb{R}} \frac{\min(1, |\xi||x-y|)}{(1+|\xi|)^{1+a}}d\xi \\
&= \int_{|\xi|\leq A} \frac{\min(1, |\xi||x-y|)}{(1+|\xi|)^{1+a}}d\xi + \int_{|\xi|>A} \frac{\min(1, |\xi||x-y|)}{(1+|\xi|)^{1+a}}d\xi \\
&:= I + II.
\end{aligned}$$

Analysing the terms separately:

$$\begin{aligned}
I &\leq \int_{|\xi|\leq A} \frac{|\xi||x-y|}{(1+|\xi|)^{1+a}}d\xi \\
&= |x-y| \int_{|\xi|\leq A} \frac{|\xi|}{(1+|\xi|)^{1+a}}d\xi \\
&\leq |x-y| \int_{|\xi|\leq A} |\xi|^{-a}d\xi \\
&= 2|x-y|\frac{A^{1-a}}{1-a}
\end{aligned}$$

and

$$\begin{aligned}
II &\leq \int_{|\xi|>A} \frac{1}{(1+|\xi|)^{1+a}}d\xi \\
&\leq \int_{|\xi|>A} \frac{1}{|\xi|^{1+a}}d\xi \\
&= 2\frac{A^{-a}}{a},
\end{aligned}$$

we see that choosing $A = |x-y|^{-1} > 0$ gives us that

$$\begin{aligned}
|f(x) - f(y)| &\lesssim I + II \\
&\leq 2|x-y|\frac{|x-y|^{a-1}}{1-a} + 2\frac{|x-y|^a}{a}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{a(1-a)}|x-y|^a \\
&= M_a|x-y|^a,
\end{aligned}
$$

where $M_a = \frac{2}{a(1-a)} > 0$. This proves the claim. $\square$

7. Complete the proof of Riesz-Thorin interpolation theorem in case where $p = \infty$ (sketch the changes needed in the proof, not all details are needed).

Sketch of the Proof. If $p = \infty$, then $p_0 = p_1 = \infty$ and we have that

$$\|Tf\|_{L^{r_0}} \leq M_0\|f\|_{L^\infty}, \quad \|Tf\|_{L^{r_1}} \leq M_1\|f\|_{L^\infty} \quad \text{for every} \quad f \in L^\infty(\mathbb{R}^d). \tag{2}$$

Now, recall the generalized Hölder interpolation theorem: if $0 < r_0 < r_1 \leq \infty, t \in (0,1)$ and $\frac{1}{r} = \frac{1-t}{r_0} + \frac{t}{r_1}$ then

$$\|f\|_{L^r} \leq \|f\|_{L^{r_0}}^{1-t}\|f\|_{L^{r_1}}^t. \tag{3}$$

Using (2) and (3) we get that

$$\|Tf\|_{L^r} \leq \|Tf\|_{L^{r_0}}^{1-t}\|Tf\|_{L^{r_1}}^t \leq M_0^{1-t}M_1^t\|f\|_{L^\infty}.$$

This completes the sketch of the proof of Riesz-Thorin interpolation theorem. $\square$

8. True or false: then there exists $\varphi \in \mathcal{S}(\mathbb{R})$ such that

$$\lim_{|x|\to\infty} e^{|x|}\varphi(x) = \infty \quad ?$$

Proof. The claim is true. Let $f(x) = \chi_{(-2,2)}(x)$ and $g \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} g(x)dx = 1$ and $\operatorname{spt} g \subset (-1,1)$. Now if $|x| \leq 1$, then define

$$
\begin{aligned}
h(x) :&= (g * f)(x) \\
&= \int_{-\infty}^{\infty} g(y)f(x-y)dy \\
&= \int_{-1}^{1} g(y)f(x-y)dy \\
&= \int_{-1}^{1} g(y)dy \\
&= \int_{\mathbb{R}} g(y)dy
\end{aligned}
$$

$$= 1.$$

Also, $h(x) = 0$ if $|x| \geq 3$ since then $g(y)f(x-y) = 0$ for every $y \in (-1,1)$.
Define

$$\varphi(x) := (1 - h(x))e^{-\frac{|x|}{2}}.$$

Then $\varphi \in C^{\infty}(\mathbb{R})$ since when $x \neq 0$ φ is a product of two smooth functions and when $|x| \leq 1$ we have that $\varphi(x) = 0$. Moreover, it's not hard to see that φ is a Schwartz function.
Hence, $\varphi \in \mathcal{S}(\mathbb{R})$ and $\lim_{|x|\to\infty} e^{|x|}\varphi(x) = \lim_{|x|\to\infty} e^{\frac{|x|}{2}} = \infty$. $\square$