Algebra II

Preliminary exercises 11

1. Recall the ideas behind the following result, which can be found in [Häsä-Rämö] and [Hungerford]: If K is a field and $P \in K[X]$ is a non-zero polynomial, then the number of roots of P is at most the degree of P.

2. Let R be a (non-trivial, commutative) ring and $P \in R[X]$. Recall that by the *polynomial function* determined by P we mean the function $f_P\colon R \to R$, $c \mapsto P(c)$, that is, we substitute the elements $c \in R$ in the polynomial. Prove that if R is finite, then there always exist distinct polynomials which determine the same polynomial function.

3. Prove item (ii) of Proposition 9.4 (3.4).

4. Denote

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Show that

$$K = \{a_0 I + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

is a subalgebra of the $\mathbb{R}$-algebra $\mathbb{C}^{2\times 2}$.

5. Familiarize yourself with the quaternion algebra presented in the lecture notes:

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

As an $\mathbb{R}$-vector space $\mathbb{H}$ is essentially $\mathbb{R}^4$. We wish to define a module homomorphism $f\colon \mathbb{H} \to \mathbb{C}^{2\times 2}$, such that

$$f(1) = I,\ f(i) = \mathbf{i},\ f(j) = \mathbf{j},\ f(k) = \mathbf{k}.$$

How do we know that such a module homomorphism exists? (You are not supposed to use the definition of a module homomorphism here.)