Algebra II

Exercise 11 (17.4.2020)

1. a) Let K be an infinite field and $P, Q \in K[X]$. Suppose also that $P(x) = Q(x)$ for every $x \in K$ (that is, P and Q determine the same polynomial function). Prove that $P = Q$.
[Hint: If the coefficient ring is a field, then the number of roots of a non-zero polynomial is at most the degree of the polynomial, see for example Häsä-Rämö, Proposition 23.12, p. 276, or Hungerford, Theorem 6.7, p. 160.]
b) Give an example of a ring R and two polynomials $P, Q \in R[X]$, such that $P = Q$, but they determine the same polynomial function $R \to R$.

2. Suppose that A is an associative and unitary R-algebra. Prove that there exists an R-algebra homomorphism $f\colon R \to A$, for which $f(1_R) = 1_A$. If R is a field and A is non-trivial (that is, $A \neq \{0\}$), prove that f is injective, and thus R can be embedded as a subalgebra of A.

3. a) Prove that every element $x \in \mathbb{H} \setminus \{0\}$ has an inverse element.
b) Let $x = x_1 i + x_2 j + x_3 k,\ y = y_1 i + y_2 j + y_3 k \in \mathbb{H}$. Verify that the product xy can be represented using the dot and cross products of the vectors $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ as follows:

$$xy = -(\bar{x} \cdot \bar{y}) + (\bar{x} \times \bar{y}) \cdot (i, j, k).$$

4. Prove that the elements

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

of the matrix algebra $\mathbb{C}^{2\times 2}$ have the same multiplication table as the quaternions i, j, k. Deduce from this that the subspace of the $\mathbb{R}$-vector space $\mathbb{C}^{2\times 2}$ generated by the unit matrix and the matrices $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is an $\mathbb{R}$-algebra, which is isomorphic with the quaternion algebra $\mathbb{H}$. Especially we see from this that $\mathbb{H}$ is associative.

5. Let R be a (non-trivial, commutative, unitary) ring. Prove that the following conditions are equivalent:

(i) R is a field.

(ii) The only ideals of R are $\{0\}$ and R.

(iii) $\{0\}$ is a maximal ideal of R.

(iv) Every non-trivial (that is, not the zero map) ring homomorphism $R \to S$ is injective (S an arbitrary ring).

6. Prove that (up to isomorphism) there exist exactly three two dimensional unitary $\mathbb{R}$-algebras.
[Hint: Identify the subspace generated by the unit element with the coefficient ring $\mathbb{R}$, as in Exercise 2. Choose 1 as one basis vector and as second basis vector some $b \notin \mathbb{R}$. Consider the multiplication table of the basis vectors. If $b^2 = x + yb$ (where $x, y \in \mathbb{R}$), define $b' = b - \frac{y}{2}$ ... We may assume that $b^2 \in \mathbb{R}$ ...]