

## FOURIER ANALYSIS II. (spring 2020)

### 5. EXERCISES (Thursday 23.4)

**NOTE:** return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

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They should be returned at the latest during the day of the 'exercise class'. The course will be evaluated based on the returned exercises.

1. Determine a fundamental solution of Laplacian in 1-dimension, i.e. find  $E \in \mathcal{S}'(\mathbf{R})$  so that  $(\frac{d}{dx})^2 = \delta_0$ .
2. Use the Poisson summation formula to prove

$$\sum_{n \in \mathbf{Z}} \frac{1}{1+n^2} = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}$$

[Recall the Fourier transform of  $f(x) = e^{-|x|}$ .]

3. (i) Suppose  $A : \mathbf{R}^d \rightarrow \mathbf{R}^d$  is an invertible linear map (we denote by  $A$  also its matrix). If  $f \in L^1(\mathbf{R}^d)$ , define  $g(x) = f(Ax)$ . Show that

$$\widehat{g}(\xi) = \frac{1}{|\det(A)|} \widehat{f}((A^{-1})^T \xi),$$

where  $(A^{-1})^T$  is the transpose of the inverse of  $A$ .

(ii) A function  $f \in L^1(\mathbf{R}^d)$  is radial if  $f(x)$  depends only on  $|x|$ . Use (i) to show that for a radial function, the Fourier transform is radial.

(iii) Show that the result in (ii) holds also for every radial  $f \in L^2(\mathbf{R}^d)$ .

4. Show that if  $E$  is a fundamental solution of the differential operator (with constant coefficients)  $P(\partial)$ , then  $E + H$  is also a fundamental solution, if  $H \in \mathcal{S}'(\mathbf{R}^d)$  satisfies  $P(\partial)H = 0$ . Verify that actually all fundamental solutions of  $P$  are obtained by this manner.
5. The Weierstrass type function

$$f(x) := \sum_{n=1}^{\infty} 2^{-n/2} \cos(2^n x)$$

can be shown not to be differentiable at any point. Show in any case that in the sense of distributions we have

$$f'(x) = - \sum_{n=1}^{\infty} 2^{n/2} \sin(2^n x) \quad !$$

6. Let  $A = \{(x, y) : x > 0, y > 0\} \cup \{(x, y) : x < 0, y < 0\} \subset \mathbf{R}^2$ .

Show that the characteristic function  $\chi_A$  is a fundamental solution for the differential operator  $P_1(\partial) = \frac{1}{2}\partial_1\partial_2$ .

7. (i) If  $0 < \gamma < d$ , show that the function

$$f_\gamma(x) = \frac{1}{|x|^\gamma}, \quad x \in \mathbf{R}^d \setminus \{0\},$$

determines a tempered distribution, by writing it as a sum of two functions, one belonging to  $L^1(\mathbf{R}^d)$  and the other to  $L^p(\mathbf{R}^d)$  for a suitable  $p > 1$ . If  $\gamma > d/2$ , show that one can choose  $p = 2$ .

(ii) Prove that

$$\widehat{f}_\gamma(\xi) = c(d, \gamma) \frac{1}{|\xi|^{d-\gamma}}$$

for some constant  $c(d, \gamma)$ .

[Hints: Consider first the case  $\gamma > d/2$  and use (i) to show  $\widehat{f}_\gamma$  is a function. Apply Problem 2 together with the scaling property  $f_\gamma(tx) = t^{-\gamma}f_\gamma(x)$ .

The case  $\gamma < d/2$  follows by inverse transform, and case  $\gamma = d/2$  by a limiting argument.]

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Try to find a fundamental solution to  $\Delta^2$  in  $\mathbf{R}^3$ .

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<sup>1</sup>These \*-exercises are extras for aficionados, not required to get full points from exercises