Exercises 5

The Deadline is Monday 20 Apr (**NOT** the 13th). Please write your name.

Here k is an algebraically closed field.

Exercise 1 (12 points)**.** Show that the set

$$GL_n(k) = \{n \times n \text{ invertible matrices with entries in } k\}$$

is a smooth affine variety of dimension n^2.

Exercise 2 (12 points)**.** Let $\varphi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be an isomorphism.

(a) Show that there exists an invertible matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2$$

such that $\varphi([x, y]) = [ax + by, cx + dy]$ [Hint: you need Exercise 1(a) from Week 2]

(b) Given two matrices A and B inducing the same isomorphim $\varphi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ as in the previous point, show that $A = \lambda B$ for some scalar $0 \neq \lambda \in k$.

Exercise 3 (12 points)**.** Prove that the product $X \times Y$ of two irreducible quasi-projective algebraic sets is irreducible. [Hint: The subsets $X \times \{y\}$ are irreducible for any $y \in Y$. Suppose $X \times Y = U_1 \cup U_2$ for some proper closed subsets U_1, U_2, consider the subsets

$$Y_i := \{y \in Y \mid X \times \{y\} \subset U_i\} \quad \text{for } i = 1, 2]$$

Exercise 4 (14 points)**.** Assume k of char. 0. Let V be a vector space of dimension 4.

(a) Let $\eta \in \bigwedge^2 V$. Prove that one and only one of the following equalities holds:

- $\eta = 0$,
- $\eta = v_1 \wedge v_2$ with v_1, v_2 linearly independent (i.e. $Span(v_1, v_2)$ has dimension 2),
- $\eta = v_1 \wedge v_2 + v_3 \wedge v_4$, with v_1, v_2, v_3, v_4 linearly independent (i.e. $Span(v_1, v_2, v_3, v_4) = V$).

(b) Deduce that η is totally decomposable if and only if $\eta \wedge \eta = 0$

(c) Deduce that the image of the Plücker embedding

$$p : \mathrm{Gr}(1, \mathbb{P}^3_k) = \mathrm{Gr}(2, 4) \hookrightarrow \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5_k$$

is a smooth quadric hypersurface, i.e. $p(\mathrm{Gr}(2, 4)) = V(F)$ with $F \in k[Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}]$ homogeneous polynomial of degree 2.