Algebra II. Exercise 10.
Solutions.

1. Let G be an Abelian group. Choose a generating set X for G (for example $X = G$ works). Define

$$f: G \to G, \quad f(x) = x.$$

By the universal property of free modules, there exists a $\mathbb{Z}$-linear map $\varphi: \mathbb{Z}^{(G)} \to G$, such that $\varphi(x) = f(x) = x$. If $x \in G$, then $x = f(x) = \varphi(x) \in \mathrm{Im}(\varphi)$, thus φ is surjective. (Here we have again identified G with a subset of $\mathbb{Z}^{(G)}$.) Since $\mathbb{Z}$-linear maps are group homomorphisms, the homomorphism theorem gives us now

$$\mathbb{Z}^{(X)}/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi) = G.$$

2. For every $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$ we have

- (B1) $(x+y) \cdot z = (x_1+y_1, ..., x_n+y_n) \cdot (z_1, ..., z_n) = (x_1+y_1)z_1 + ... + (x_n+y_n)z_n = x_1z_1 + ... + x_nz_n + y_1z_1 + ... + y_nz_n = x \cdot z + y \cdot z$.
- (B2) $x \cdot (y+z) = x \cdot y + x \cdot z$ similarly.
- (B3) $(ax) \cdot z = (ax_1, ..., ax_n) \cdot (z_1, ..., z_n) = ax_1z_1 + ... + ax_nz_n = a(x_1z_1 + ... + x_nz_n) = a(x \cdot z)$.
- (B4) $x \cdot (az) = (x_1, ..., x_n) \cdot (az_1, ..., az_n) = x_1az_1 + ... + x_naz_n = a(x_1z_1 + ... + x_nz_n) = a(x \cdot z)$.

3.

$$\eta: M \times N \to M \otimes N$$

$$(x, y) \mapsto x \otimes y$$

Let $x, x' \in M$, $y \in N$, $a \in R$.
Condition (B1): We should prove that $(x+x') \otimes y = (x \otimes y) + (x' \otimes y)$. Since the element $(x+x', y) - (x, y) - (x', y) \in D$, we have that

$$0 = [(x+x', y) - (x, y) - (x', y)] = [(x+x', y)] - [(x, y)] - [(x', y)]$$

$$= (x + x') \otimes y - (x \otimes y) - (x' \otimes y),$$

from which it follows that $(x+x')\otimes y = (x\otimes y)+(x'\otimes y)$, that is, $\eta(x+x', y) = \eta(x, y) + \eta(x', y)$.
Condition (B2) analogously with respect to the 2nd coordinate.
Condition (B3): Since the element $(ax, y) - a(x, y) \in D$, we have that

$$0 = [(ax, y)-a(x, y)] = [(ax, y)]-[a(x, y)] = [(ax, y)]-a[(x, y)] = (ax)\otimes y-a(x\otimes y),$$

from which it follows that $(ax) \otimes y = a(x \otimes y)$, that is, $\eta(ax, y) = a\eta(x, y)$.
Condition (B4) analogously with respect to the 2nd coordinate.

4. Let x be a generator of the module $\mathbb{Q} \otimes_{\mathbb{Z}} M$, that is, x is of the form $a \otimes m$, where $a \in \mathbb{Q}$, $m \in M$. Since the group M is finite, we know that the order $\mathrm{ord}(m)$ is finite; suppose $\mathrm{ord}(m) = k$. Now

$$a \otimes m = \frac{ak}{k} \otimes m = k(\frac{a}{k} \otimes m) = \frac{a}{k} \otimes km = \frac{a}{k} \otimes 0 = 0.(\frac{a}{k} \otimes 0) = 0.$$

Thus every generator is the zero element, from which the claim follows.

5. Suppose that $\varphi\colon M \to M'$ is an R-linear isomorphism. Consider the composite map

$$f = \eta' \circ (\varphi \times \mathrm{id})\colon M \times P \to M' \times P \to M' \otimes P$$

$$(m, p) \mapsto (\varphi(m), p) \mapsto \varphi(m) \otimes p.$$

We verify that f is R-bilinear:

(B1) : $$f((m + m'), p) = \varphi(m + m') \otimes p = (\varphi(m) + \varphi(m')) \otimes p$$

$$= \varphi(m) \otimes p + \varphi(m') \otimes p = f(m, p) + f(m', p);$$

(B2) :
$f(m, p + p') = \varphi(m) \otimes (p + p') = \varphi(m) \otimes p + \varphi(m) \otimes p' = f(m, p) + f(m, p');$

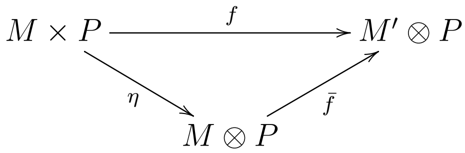
(B3) : $$f(rm, p) = \varphi(rm) \otimes p = (r\varphi(m)) \otimes p = r(\varphi(m) \otimes p) = rf(m, p);$$

(B4) : $$f(m, rp) = \varphi(m) \otimes (rp) = r(\varphi(m) \otimes p) = rf(m, p).$$

Now by the univerasl property of the tensor product we have that f induces an R-linear map $\bar{f}\colon M \otimes P \to M' \otimes P$, for which

$$\bar{f}(m \otimes p) = \varphi(m) \otimes p$$

for every $m \in M, p \in P$, that is, the diagram



commutes.
By applying the corresponding construction to the inverse map φ^{-1}, we obtain an R-linear map $\bar{g}\colon M' \otimes P \to M \otimes P$, for which

$$\bar{g}(m' \otimes p) = \varphi^{-1}(m') \otimes p$$

for every $m' \in M', p \in P$. Now

$$\bar{g} \circ \bar{f}(m \otimes p) = \bar{g}(\varphi(m) \otimes p) = \varphi^{-1}(\varphi(m)) \otimes p = m \otimes p,$$

and similarly

$$\bar{f} \circ \bar{g}(m' \otimes p) = m' \otimes p.$$

Since the elements $m \otimes p$ generate the module $M \otimes P$ and $\bar{f}$ and $\bar{g}$ are linear, it follows that $\bar{g} \circ \bar{f} = \mathrm{id}$ and similarly $\bar{f} \circ \bar{g} = \mathrm{id}$. Thus $\bar{f}$ is an R-linear isomorphism.

6. a) Define $f\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, $f(x, y) = xy$. The map f on $\mathbb{Z}$-bilinear, for example

(B1) : $$f(x + x', y) = (x + x')y = xy + x'y = f(x, y) + f(x', y);$$

(B3) : $$f(ax, y) = (ax)y = a(xy) = af(x, y).$$

Thus by the universal property there exists a $\mathbb{Z}$-linear map $\varphi\colon \mathbb{Q} \otimes \mathbb{Q} \to \mathbb{Q}$, for which $x \otimes y \mapsto xy$.
b) Let $y \in \mathbb{Q} \otimes \mathbb{Q}$, then

$$y = \sum_i \left(\frac{a_i}{b_i} \otimes \frac{c_i}{d_i}\right) = \sum_i \frac{a_i d_i}{b_i d_i} \otimes \frac{c_i}{d_i} = \sum_i \frac{a_i}{b_i d_i} \otimes \frac{c_i d_i}{d_i}$$

$$= \sum_i \frac{a_i}{b_i d_i} \otimes c_i = \sum_i \frac{a_i c_i}{b_i d_i} \otimes 1 = \left(\sum_i \frac{a_i c_i}{b_i d_i} \right) \otimes 1,$$

where $a_i, b_i, c_i, d_i \in \mathbb{Z}$. Thus

$$y = \psi \left(\sum_i \frac{a_i c_i}{b_i d_i} \right),$$

and we see that ψ is surjective.

Finally we prove that φ and ψ are inverse maps of each other:

$\psi \circ \varphi = \mathrm{id}$: Let $y \in \mathbb{Q} \otimes \mathbb{Q}$. By the above there exists $a \in \mathbb{Q}$, for which $y = a \otimes 1$. Thus $\psi(\varphi(y)) = \psi(\varphi(a \otimes 1)) = \psi(a) = a \otimes 1 = y$.

$\varphi \circ \psi = \mathrm{id}$: Let $a \in \mathbb{Q}$. Now $\varphi(\psi(a)) = \varphi(a \otimes 1) = a \cdot 1 = a$.

Thus φ is an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$.