FOURIER ANALYSIS II. (spring 2020)

4. EXERCISES (Thursday 16.4)

NOTE: return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

stefanos.lappas@helsinki.fi

They should be returned at the latest during the day of the 'exercise class'. The course will be evaluated based on the returned exercises.

1. Assume that the sequence of measurable functions f_n is uniformly bounded, i.e. $|f_n(x)| \le C$ for all $x \in \mathbf{R}^n$ and $n \ge 1$, and it converges at almost every point:

$$\lim f_n(x) = g(x) \qquad \text{for almost every } \; x \in \mathbf{R}^d.$$

Show that the $f_n \to g$ in the sense of distributions.

2. Is the function $x^2 \sin(x)$ the Fourier transform of a distribution ? If so, determine the distribution.

[Hint: Use first Euler's formula to see what is the Fourier transform of $\sin x$ then recall how Fourier transform (of distributions) behaves under differentiation.]

3. (i) Let $g \in \mathcal{S}(\mathbf{R})$. Show that in the metric of the space $\mathcal{S}(\mathbf{R})$ it holds that $f_\varepsilon(x) \to f'(x)$ as $\varepsilon \to 0^+$, where $f_\varepsilon(x) := \varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big)$.

(ii) Use part (i) to verify that in a similar manner for any $f \in L^1(\mathbf{R})$

$$\varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big) \to \frac{d}{dx} f \quad \text{as} \quad \varepsilon \to 0,$$

where $\frac{d}{dx} f$ is the derivative of f in the sense of distributions.

4. (i) Show that $< \mathcal{F}^{-1}T, g > = < T, \mathcal{F}^{-1}g >$ for all $T \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$.

(ii) Verify that $\mathcal{F}^4 \lambda = (2\pi)^{2d} \lambda$ for any $\lambda \in \mathcal{S}'(\mathbf{R}^d)$.

5. Let $K \in L^1$ with $\int_{\mathbf{R}^d} K(x)dx = 1$ and set $K_\varepsilon(x) := \varepsilon^{-d} K(x/\varepsilon)$ for any $\varepsilon > 0$. Prove that in the sense of distributions

$$\lim_{\varepsilon \to 0^+} K_\varepsilon = \delta_0.$$

[Suggestion: do it directly, or prove the same convergence for the Fourier transforms!]

6. Show that $f(x) = \log|x| \in \mathcal{S}'(\mathbf{R})$ and that the distributional derivative of f is

$$\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$$

[Compute first the derivative of $\chi_{|x|\geq\varepsilon}\log(|x|)$. Use the fact that for test functions $|g(u) - g(-u)| \leq C|u|$.]

7. Let $\psi \in C_0^\infty(\mathbf{R})$. Determine the Fourier transform of the distribution λ, where

$$\langle\lambda, g\rangle := \int_{\mathbf{R}} \psi(u)g(u,0)du \qquad \text{for all} \quad g \in \mathcal{S}(\mathbf{R}^2).$$

8*[1] **(i)** Define $h(x) := \int_0^x \frac{\sin t}{t}dt$. Show that $h : [0,\infty) \to \mathbf{R}$ is a bounded function.

(ii) Determine $A := \lim_{x\to\infty} h(x) = \int_0^\infty \frac{\sin t}{t}dt$ by observing that formally

$$A = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin t}{t}dt = 2\pi\frac{1}{2}\mathcal{F}^{-1}\Big(\frac{\sin x}{x}\Big)(0),$$

and trying to make this somehow rigorous.

[part (i) is not difficult!]

[1] These $*$-exercises are extras for afficinadoes, not required to get full points from exercises