Fourier Analysis II

Spring 2020

Homework 4

Exercise session: Thu 16 April; Stefanos Lappas, stefanos.lappas@helsinki.fi.

1. Assume that the sequence of measurable functions f_n is uniformly bounded, i.e. $|f_n(x)| \leq C$ for all $x \in \mathbb{R}^n$ and $n \geq 1$, and it converges at almost every point:

$$\lim f_n(x) = g(x) \qquad \text{for almost every } \ x \in \mathbb{R}^d.$$

Show that the $f_n \to g$ in the sense of distributions.

Proof. Fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We know that

$$\int_{\mathbb{R}^d} |f_n(x)\varphi(x)|dx \leq \int_{\mathbb{R}^d} C|\varphi(x)|dx < \infty,$$

so we can use the dominated convergence theorem to see that

$$\lim_{n\to\infty} \langle f_n, \varphi\rangle = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(x)\varphi(x)dx = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(x)\varphi(x)dx = \int_{\mathbb{R}^d} g(x)\varphi(x)dx = \langle g, \varphi\rangle.$$

This shows that $f_n \to g$ in the sense of distributions. □

2. Is the function $x^2 \sin(x)$ the Fourier transform of a distribution ? If so, determine the distribution.

Proof. The claim is true. We may compute that

$$\begin{aligned}
\langle x^2 \sin(x), g\rangle &= \int_{\mathbb{R}} x^2 \sin(x) g(x) dx \\
&= \int_{\mathbb{R}} \frac{1}{2i}(e^{ix} - e^{-ix}) x^2 g(x) dx \\
&= \frac{1}{2i}\left(\widehat{(x^2 g)}(-1) - \widehat{(x^2 g)}(1)\right) \\
&= \frac{i}{2}\left(\frac{d^2\widehat{g}}{dx^2}(-1) - \frac{d^2\widehat{g}}{dx^2}(1)\right)
\end{aligned}$$

$$= \frac{i}{2}\langle \delta''_{-1} - \delta''_1, \widehat{g}\rangle$$
$$= \frac{i}{2}\Big\langle (\widehat{\delta''_{-1} - \delta''_1}), g\Big\rangle, \quad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}).$$

Thus $x^2 \sin(x) = \widehat{T}$, where $T := \frac{i}{2}(\delta''_{-1} - \delta''_1)$. Here of course δ is the Dirac delta distribution. □

3. (i) Let $f \in \mathcal{S}(\mathbb{R})$. Show that in the metric of the space $\mathcal{S}(\mathbb{R})$ it holds that $f_\varepsilon(x) \to f'(x)$ as $\varepsilon \to 0^+$, where $f_\varepsilon(x) := \varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big)$.

(ii) Use part (i) to verify that in a similar manner for any $f \in L^1(\mathbb{R})$

$$\varepsilon^{-1}\big(f(x+\varepsilon) - f(x)\big) \to \frac{d}{dx}f \quad \text{as} \quad \varepsilon \to 0,$$

where $\frac{d}{dx}f$ is the derivative of f in the sense of distributions.

Proof. **(i)** Fix N, and note that

$$p_N(f_\varepsilon - f') = \sup_{n\le N}\sup_{x\in\mathbb{R}}(1+|x|^2)^N\left|\frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) - f^{(n+1)}(x)\right|.$$

We now estimate the expression inside. By the mean value theorem there exists $y \in [x, x+\varepsilon]$ so that

$$\frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) = f^{(n+1)}(y).$$

Similarly there exists $z \in [x, y]$ such that

$$f^{(n+1)}(y) - f^{(n+1)}(x) = f^{(n+2)}(z)(y-x).$$

Thus

$$\left|\frac{1}{\varepsilon}(f^{(n)}(x+\varepsilon) - f^{(n)}(x)) - f^{(n+1)}(x)\right| = |f^{(n+2)}(z)||y-x| \le \varepsilon|f^{(n+2)}(z)|.$$

Note also that since $|z-x| \le \varepsilon$, we have for sufficiently small ε that

$$(1+|x|^2)^N \le 2^N(1+|z|^2)^N.$$

The exact value of the constant 2^N here doesn't really matter, but the proof of this estimate can be done as follows:

$$|x| \le |z| + \varepsilon \Rightarrow |x|^2 \le |z|^2 + 2\varepsilon|z| + \varepsilon \Rightarrow 1 + |x|^2 \le 1 + 2\varepsilon + (1+\varepsilon)|z|^2 \le 2(1+|z|^2).$$

By combining everything we finally get that

$$\begin{aligned} p_N(f_\varepsilon - f') &\leq \sup_{n\leq N}\sup_{x\in\mathbb{R}} 2^N(1+|z|^2)^N\varepsilon|f^{(n+2)}(z)| \\ &\leq \varepsilon 2^N p_{N+2}(f). \end{aligned}$$

This shows that $f_\varepsilon \to f'$ in the topology of $\mathcal{S}(\mathbb{R})$ as $\varepsilon \to 0$.
(ii) Let $g \in \mathcal{S}(\mathbb{R})$. We compute that

$$\begin{aligned} \langle f_\varepsilon, g\rangle &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}(f(x+\varepsilon)-f(x))g(x)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\varepsilon}(g(x-\varepsilon)-g(x))f(x)dx \\ &= \int_{-\infty}^{\infty} g_{-\varepsilon}(x)f(x)dx, \end{aligned}$$

where we used change of variables. Applying part (**i**) for the function $h(x) := g(-x)$ we have that the functions $g_{-\varepsilon}$ converge uniformly to $-g'$ as $\varepsilon \to 0$ so we can use dominated convergence theorem to conclude that

$$\lim_{\varepsilon\to 0}\langle f_\varepsilon, g\rangle = -\int_{-\infty}^{\infty} g'(x)f(x)dx = -\langle f, g'\rangle = \langle f', g\rangle.$$

This shows that $\lim_{\varepsilon\to 0} f_\varepsilon = f'$ in $\mathcal{S}'(\mathbb{R})$. □

4. (i) Show that $< \mathcal{F}^{-1}T, g > = < T, \mathcal{F}^{-1}g >$ for all $T \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Verify that $\mathcal{F}^4\lambda = (2\pi)^{2d}\lambda$ for any $\lambda \in \mathcal{S}'(\mathbb{R}^d)$.

Proof. **(i)** Let us show that defining the inverse Fourier transform $\mathcal{F}^{-1}$ on $\mathcal{S}'$ by

$$\langle \mathcal{F}^{-1}T, g\rangle = \langle T, \mathcal{F}^{-1}g\rangle, \quad \text{for all} \quad T \in \mathcal{S}'(\mathbb{R}^d) \quad \text{and} \quad g \in \mathcal{S}(\mathbb{R}^d),$$

actually gives an inverse of the Fourier transform. This is easily seen since

$$\langle \mathcal{F}^{-1}\widehat{T}, g\rangle = \langle \widehat{T}, \mathcal{F}^{-1}g\rangle = \langle T, \mathcal{F}\mathcal{F}^{-1}g\rangle = \langle T, g\rangle$$

and

$$\langle \mathcal{F}\mathcal{F}^{-1}T, g\rangle = \langle \mathcal{F}^{-1}T, \widehat{g}\rangle = \langle T, \mathcal{F}^{-1}\widehat{g}\rangle = \langle T, g\rangle.$$

(ii) Recall that for any $g \in \mathcal{S}(\mathbb{R}^d)$ we have $(\mathcal{F}^2 g)(x) = (2\pi)^d g(-x)$, which implies that $(\mathcal{F}^4 g)(x) = (2\pi)^{2d} g(x)$. Now we simply compute

$$\langle \mathcal{F}^4 \lambda, g \rangle = \langle \lambda, \mathcal{F}^4 g \rangle = \langle \lambda, (2\pi)^{2d} g \rangle = \langle (2\pi)^{2d} \lambda, g \rangle.$$

□

5. Let $K \in L^1$ with $\int_{\mathbb{R}^d} K(x)dx = 1$ and set $K_\varepsilon(x) := \varepsilon^{-d} K(x/\varepsilon)$ for any $\varepsilon > 0$. Prove that in the sense of distributions

$$\lim_{\varepsilon \to 0^+} K_\varepsilon = \delta_0.$$

Proof. Fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\eta > 0$. As φ is continuous, there is some $\delta > 0$ such that $|\varphi(x) - \varphi(0)| < \eta$ whenever $|x| < \delta$.
Now we compute

$$\begin{aligned}
|\langle K_\varepsilon, \varphi \rangle - \langle \delta_0, \varphi \rangle| &= \left| \int_{\mathbb{R}^d} K_\varepsilon(x)\varphi(x)dx - \varphi(0) \right| \\
&= \left| \int_{\mathbb{R}^d} K_\varepsilon(x)(\varphi(x) - \varphi(0))dx \right| \\
&\leq \int_{|x|<\delta} |K_\varepsilon(x)||\varphi(x) - \varphi(0)|dx + \int_{|x|\geq\delta} |K_\varepsilon(x)||\varphi(x) - \varphi(0)|dx \\
&\leq \eta \int_{|x|<\delta} |K_\varepsilon(x)|dx + 2p_0(\varphi) \int_{|x|\geq\delta} |K_\varepsilon(x)|dx \\
&\leq \eta \int_{\mathbb{R}^d} |K(x)|dx + 2p_0(\varphi) \int_{|x|\geq\delta/\varepsilon} |K(x)|dx,
\end{aligned}$$

where in the last step we used the change of variables. The first term is bounded by $\eta \|K\|_{L^1(\mathbb{R}^d)}$ and the second term converges to zero by the dominated convergence theorem. This means that

$$\limsup_{\varepsilon \to 0^+} |\langle K_\varepsilon, \varphi \rangle - \langle \delta_0, \varphi \rangle| \leq \eta \|K\|_{L^1(\mathbb{R}^d)}.$$

As η was arbitrary, the claim follows. □

6. Show that $f(x) = \log|x| \in \mathcal{S}'(\mathbb{R})$ and that the distributional derivative of f is

$$\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$$

Proof. The function $\log|x|$ is L^1-integrable around $x = 0$ and grows slower than a polynomial as $|x| \to \infty$. This easily shows that it defines a tempered distribution on $\mathbb{R}$. We now compute that

$$\begin{aligned}
\Big\langle \frac{d}{dx}\log|x|, g\Big\rangle &= -\Big\langle \log|x|, \frac{d}{dx}g\Big\rangle \\
&= -\int_{-\infty}^{\infty} \log|x| g'(x)dx \\
&= -\int_{-\infty}^{0} \log(-x)g'(x)dx - \int_{0}^{\infty} \log(x)g'(x)dx \\
&= -\lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon} \log(-x)g'(x)dx - \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} \log(x)g'(x)dx \\
&= \lim_{\varepsilon\to 0}(-\log(\varepsilon)g(-\varepsilon) + \log(1/\varepsilon)g(-1/\varepsilon) - \log(1/\varepsilon)g(1/\varepsilon) + \log(\varepsilon)g(\varepsilon)) \\
&\qquad + \lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon} \frac{1}{x}g(x)dx + \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} \frac{1}{x}g(x)dx \\
&= \Big\langle \mathbf{p.v.}\frac{1}{x}, g\Big\rangle, \quad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}),
\end{aligned}$$

where we have used integration by parts here, and dominated convergence theorem to conclude that

$$\int_{-\infty}^{0} \log(-x)g'(x)dx = \lim_{\varepsilon\to 0}\int_{-1/\varepsilon}^{-\varepsilon} \log(-x)g'(x)dx$$

and

$$\int_{0}^{\infty} \log(x)g'(x)dx = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1/\varepsilon} \log(x)g'(x)dx.$$

Additionally, since g is in $\mathcal{S}(\mathbb{R})$ we were able to conclude that

$$\lim_{\varepsilon\to 0}\log(1/\varepsilon)g(-1/\varepsilon) = \lim_{\varepsilon\to 0}\log(1/\varepsilon)g(1/\varepsilon) = 0$$

and

$$\lim_{\varepsilon\to 0}(\log(\varepsilon)g(\varepsilon) - \log(\varepsilon)g(-\varepsilon)) = \lim_{\varepsilon\to 0}\varepsilon\log(\varepsilon)\frac{g(\varepsilon) - g(-\varepsilon)}{\varepsilon} = 0.$$

Thus $\frac{d}{dx}(\log|x|) = \mathbf{p.v.}\frac{1}{x}$. □

7. Let $\psi \in C_0^\infty(\mathbb{R})$. Determine the Fourier transform of the distribution λ, where

$$\langle \lambda, g\rangle := \int_{\mathbb{R}} \psi(u)g(u,0)du \qquad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^2).$$

Proof. Using the definition of the Fourier transform for distributions and Fubini's theorem, we have that

$$\begin{aligned}
\langle \widehat{\lambda}, g\rangle &= \langle \lambda, \widehat{g}\rangle \\
&= \int_{\mathbb{R}} \psi(u)\widehat{g}(u,0)du \\
&= \int_{\mathbb{R}} \psi(u) \int_{\mathbb{R}^2} e^{-iux} g(x,y)dxdydu \\
&= \int_{\mathbb{R}^2} g(x,y) \int_{\mathbb{R}} \psi(u)e^{-iux}dudxdy \\
&= \int_{\mathbb{R}^2} g(x,y)\widehat{\psi}(x)dxdy, \quad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^2).
\end{aligned}$$

We have shown that $\widehat{\lambda}$ is a distribution with

$$\widehat{\lambda}(x,y) = \widehat{\psi}(x).$$

□

8. (i) Define $h(x) := \int_0^x \frac{\sin t}{t} dt$. Show that $h : [0, \infty) \to \mathbb{R}$ is a bounded function.

(ii) Determine $A := \lim_{x\to\infty} h(x) = \int_0^\infty \frac{\sin t}{t} dt$ by observing that formally

$$A = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin t}{t} dt = 2\pi \frac{1}{2} \mathcal{F}^{-1}(\frac{\sin x}{x})(0),$$

and trying to make this somehow rigorous.

Proof. **(i)** First of all, we can extend the function h by continuity at zero, i.e., $h(0) = 1$. We see that h has extrema at points $x = n\pi$ for any positive integer n. Now, consider the sequence

$$a_n = h(n\pi) - h((n-1)\pi).$$

Then,

$$|a_n| = \left| \int_0^{n\pi} \frac{\sin t}{t} dt - \int_0^{(n-1)\pi} \frac{\sin t}{t} dt \right| \le \left| \int_{(n-1)\pi}^{n\pi} \frac{\sin t}{t} dt \right| \le \frac{n\pi - (n-1)\pi}{(n-1)\pi} = \frac{1}{n-1},$$

so $a_n \to 0$ as $n \to \infty$. We also see that the sequence a_n is decreasing:

$$\begin{aligned} a_n - a_{n+1} &= \int_0^{n\pi} \frac{\sin t}{t} dt - \int_0^{(n-1)\pi} \frac{\sin t}{t} dt - \left(\int_0^{(n+1)\pi} \frac{\sin t}{t} dt - \int_0^{n\pi} \frac{\sin t}{t} dt \right) \\ &= \int_{(n-1)\pi}^{n\pi} \frac{\sin t}{t} dt - \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t} dt \\ &= \int_0^{\pi} \sin y \left(\frac{1}{n\pi - \pi + y} - \frac{1}{n\pi + y} \right) dy > 0, \end{aligned}$$

where in the last step we used change of variables for both of the integrals.
We therefore know that there exists a limit of extreme values $\lim_{n\to\infty} a_n$, so the function has a limit at infinity and is therefore bounded.
(ii) Let the Heaviside step function:

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

By exercise 1 of exercise set 1 we have that:

$$\widehat{f}(\xi) = \begin{cases} \frac{2\sin(\xi)}{\xi}, & \text{if } \xi \neq 0 \\ 2, & \text{if } \xi = 0. \end{cases}$$

Hence,

$$f(0) = \mathcal{F}^{-1}\left(\frac{2\sin x}{x} \right)(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin \xi}{\xi} d\xi.$$

Since $f(0) = 1$ and the function $\frac{\sin(\xi)}{\xi}$ is even over the symmetric interval $(-\infty, \infty)$, we conclude that

$$A = \int_0^{\infty} \frac{\sin \xi}{\xi} d\xi = \frac{\pi}{2}.$$

□