Algebra II

Preliminary exercises 12

1. Recall the basic concepts of divisibility in the case of the integers. For example, what are the units (yksiköt), associates (liittoalkiot), irreducible elements (jaottomat alkiot), and prime elements (alkualkiot) in this case? Recall also Euclid's lemma and its' proof.

2. Show that all greatest common divisors of two elements are associates (now we are in the general case, R is any integral domain).

3. Recall the following result: Let K be a field. An element $c \in K$ is a root of a polynomial $P \in K[X]$, if and only if P is divisible by the polynomial $X - c$.

4. Consider $\mathbb{R}[X]$ (or more generally $K[X]$ with K a field). In general one cannot say that if a polynomial doesn't have roots, then it is irreducible. Give an example. However we can say that if the degree of a polynomial is at most three, and it doesn't have roots, then it is irreducible. Why?

5. Investigate the following polynomials in $\mathbb{Q}[X]$. Which are irreducible?

a) $X^3 + 2X^2 + X - 4$

b) $X^3 + 2X^2 + X - 5$

c) $-3X^5 + 6X^3 - 2$ [Hint: Eisenstein]

d) $2X^3 + 5X^2 + 6X + 24$ [Hint: Investigate the coefficients modulo 5]