Algebra II

Exercise 12 (27.4.2020)

1. (Recall exercises 2 and 6 of last week). Define multiplication in the $\mathbb{R}$-module $\mathbb{R}^2$ by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1, a_2 b_2).$$

a) With this multiplication $\mathbb{R}^2$ is an associative, commutative, unitary $\mathbb{R}$-algebra. Prove that it is isomorphic to the "$b^2 > 0$" case of exercise 6 last week.

b) With this multiplication $\mathbb{R}^2$ is also a ring. Is it a field?

[Additional information. The complex numbers correspond to the case "$b^2 < 0$". A model for the case "$b^2 = 0$" can be obtained as follows: Consider the set of polynomials

$$A = \{a + bX \mid a, b \in \mathbb{R}\}.$$

It is in a natural way an $\mathbb{R}$-module, isomorphic with $\mathbb{R}^2$. We define multiplication in A in such a way that we first multiply the polynomials normally, but then forget the second order term:

$$(a + bX) \cdot (c + dX) = (a + c) + (ad + bc)X.$$

With this multiplication A is a two dimensional $\mathbb{R}$-algebra and "$b^2 = 0$".]

2. Suppose that K is a field. Prove the following claims:

a) If $\mathrm{char}(K) = p > 0$, then p is a prime number.

b) If $\mathrm{char}(K) = p > 0$, then every subfield of K contains a subfield which is isomorphic with the field $\mathbb{F}_p$.

c) If $\mathrm{char}(K) = 0$, then every subfield of K contains a subfield which is isomorphic with the field $\mathbb{Q}$.

3. Prove Lemma 11.1 (5.1). Suppose that R is an integral domain and $a, b \in R$.

a) The elements a and b are associates, if and only if $a = bc$ for some element c, which is a unit.

b) If $a, b \in R \setminus \{0\}$ are associates and $a = bc$, then c is a unit.

c) All units are associates of each other.

4. Prove that in the ring

$$\mathbb{Z}[i\sqrt{5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\}$$

the number 2 is an irreducible (jaoton) number, which divides the product $(1 + i\sqrt{5})(1 - i\sqrt{5})$, but it divides neither of the factors. Deduce from this that $\mathbb{Z}[i\sqrt{5}]$ is not a unique factorization domain (tekijöihinjakorengas).

[Hint: If 2 were reducible (that is, not irreducible), then $2 = (a + bi\sqrt{5})(c + di\sqrt{5})$, $a, b, c, d \in \mathbb{Z}$. Investigate the modules of the complex numbers appearing on different sides of this equation.]

5. Suppose that R is a principal ideal domain (pääideaalirengas) and $0 \neq a \in R$. Prove that the following conditions are equivalent:

(1) a is a prime element.

(2) a is irreducible.

(3) $\langle a \rangle$ is a maximal ideal.

6. Prove that a polynomial $X^3 + aX^2 + bX + 1 \in \mathbb{Z}$ is reducible, if and only if $a = b$ or $a + b = -2$.