Exercises 6

If you can't solve a point, you can still use its statement for solving the next one.
Here k is an algebraically closed field.

Example 1 (15 points)**.** The set of $n \times n$-matrices with determinant equal to one

$$SL_n := \{A \in \mathbb{A}_k^{n^2} \mid \det(A) = 1\} = V(\det -1) \subset \mathbb{A}_k^{n^2}$$

is an affine algebraic set. We denote by $k[\ldots, x_{i,j}, \ldots]$ the polynomial ring of $\mathbb{A}_k^{n^2}$.

(a) Fix a matrix/point $A \in SL_n$. For any $1 \leq i, j \leq n$, let $A_{i,j}$ the sub-$(n-1) \times (n-1)$-matrix of A obtained by removing the ith row and the jth column. Show that

$$\left(\frac{\partial \det}{\partial x_{i,j}}\right)(A) = (-1)^{i+j} \det(A_{i,j}).$$

Here $\det(A_{i,j})$ on the right-hand side is the determinant of the $(n-1) \times (n-1)$-matrix $A_{i,j}$.

(b) Let $I \in SL_n$ be the identity matrix. Deduce that $T_I SL_n$ is the space of matrices with trace equal to zero, i.e.

$$\mathbb{A}_k^{n^2} \supset T_I SL_n = \{A \in \mathbb{A}_k^{n^2} \mid tr(A) = 0\} = V(tr) =: \mathfrak{sl}_n$$

where $tr = y_{1,1} + \ldots + y_{n,n}$

(c) Deduce that SL_n is smooth at I.

(d) Let $A \in SL_n$ be a point. Explain why the morphism

$$\begin{array}{ccc} SL_n & \xrightarrow{\varphi_A} & SL_n \\ B & \mapsto & A \cdot B \quad \text{product of matrices} \end{array}$$

is a regular isomorphism.

(e) Deduce that SL_n is smooth.

Example 2 (10 pts)**.** Let X and Y be quasi-projective algebraic sets.

(a) Show that $T_{x,y}(X \times Y) = T_x X \oplus T_y Y$ for any $(x, y) \in X \times Y$.

(b) Deduce that $\dim(X \times Y) = \dim X + \dim Y$.

Example 3 (15 points)**.** Let $Z \subset \mathbb{P}_k^n$ be a projective algebraic set and $p = [p_0, \ldots, p_n] \in \mathbb{P}_k^n \setminus Z$.
For any $z \in Z$, we denote by $L_{p,z} \subset \mathbb{P}_k^n$ the unique line containing z and p. Consider the set

$$J(p, Z) = \bigcup_{z \in Z} L_{p,z} \subset \mathbb{P}_k^n$$

Consider the regular morphism

$$\begin{array}{ccc} \mathbb{P}_k^1 \times Z & \xrightarrow{j} & \mathbb{P}_k^n \\ ([X, Y], [z_0, \ldots, z_n]) & \mapsto & [X z_0 + Y p_0, \ldots, X z_n + Y p_n] \end{array}$$

(a) Show that the image of j is $J(p, Z)$.

(b) Deduce that $J(p, Z)$ is a projective algebraic subset of $\mathbb{P}_k^n$.

(c) Show that if Z is irreducible, then $J(p, Z)$ is irreducible.

(d) Show that $\dim J(p, Z) = \dim Z + 1$.

Example 4 (10 points)**.** Show that

(a) if Z is a projective variety and an affine variety, then Z is a point (i.e. zero-dimensional irreducible algebraic set).

(b) the algebraic set $\mathbb{A}_k^m \times \mathbb{P}_k^n$ is not affine and is not projective.