Fourier Analysis II

Spring 2020

Homework 5

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1. Determine a fundamental solution of Laplacian in 1-dimension, i.e. find $E \in \mathcal{S}'(\mathbb{R})$ so that $\left(\frac{d}{dx}\right)^2 E = \delta_0$.

Proof. Let $E(x) = \frac{|x|}{2}$. For any $\varphi \in \mathcal{S}(\mathbb{R})$ we compute that

$$\begin{aligned}
\left\langle \frac{d}{dx} E(x), \varphi \right\rangle &= -\langle E(x), \varphi' \rangle \\
&= -\int_{\mathbb{R}} \frac{|x|}{2} \varphi'(x) dx \\
&= -\frac{1}{2} \int_0^\infty x\varphi'(x) dx + \frac{1}{2} \int_{-\infty}^0 x\varphi'(x) dx \\
&= \frac{1}{2} \int_0^\infty \varphi(x) dx - \frac{1}{2} \int_{-\infty}^0 \varphi(x) dx \\
&= \left\langle \frac{\operatorname{sgn}(x)}{2}, \varphi \right\rangle.
\end{aligned}$$

Hence $\frac{d}{dx} E(x) = \frac{\operatorname{sgn}(x)}{2}$. Furthermore,

$$\begin{aligned}
\left\langle \frac{d^2}{dx^2} E(x), \varphi \right\rangle &= \left\langle \frac{d}{dx} E(x), \varphi' \right\rangle \\
&= \left\langle \frac{\operatorname{sgn}(x)}{2}, \varphi' \right\rangle \\
&= \int_{\mathbb{R}} \frac{\operatorname{sgn}(x)}{2} \varphi'(x) dx \\
&= \frac{1}{2} \int_{-\infty}^0 -\varphi'(x) dx + \frac{1}{2} \int_0^\infty \varphi'(x) dx \\
&= \frac{\varphi(0)}{2} + \frac{\varphi(0)}{2} \\
&= \varphi(0) \\
&= \langle \delta_0, \varphi \rangle.
\end{aligned}$$

Thus, $\frac{d^2}{dx^2}E = \delta_0$. It follows that $E(x) = \frac{|x|}{2}$ is a fundamental solution of Laplacian in 1-dimension. □

2. Use the Poisson summation formula to prove

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi\frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

Proof. Let us recall from exercise 3 of exercise set 1 that if $f(x) = e^{-|x|}$, then

$$\widehat{f}(\xi) = \frac{2}{1+\xi^2}.$$

By the Fourier inversion formula we have also that

$$\mathcal{F}(\widehat{f})(x) = 2\pi e^{-|x|}.$$

We want to apply the Poisson summation formula on the function $\widehat{f}$. We must check that

$$|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-1-\epsilon} \quad \text{and} \quad |\mathcal{F}(\widehat{f})(x)| \leq C(1+|\xi|)^{-1-\epsilon}$$

for some constants $C, \epsilon > 0$. The second condition holds for every $\epsilon > 0$ since the exponential function grows faster than any polynomial. The first condition holds for $\epsilon = 1$ because of the estimate

$$|\widehat{f}(\xi)| = \frac{2}{1+\xi^2} \leq \frac{4}{(1+|\xi|)^2}.$$

Thus we can apply the Poisson summation formula to get that

$$\begin{aligned}
2\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \sum_{n\in\mathbb{Z}} \widehat{f}(n) &= \sum_{n\in\mathbb{Z}} \mathcal{F}(\widehat{f})(2\pi n) \\
&= \sum_{n\in\mathbb{Z}} 2\pi e^{-2\pi|n|} \\
&= 2\pi + 4\pi \sum_{n=1}^{\infty} e^{-2\pi n} \\
&= 2\pi + 4\pi \frac{e^{-2\pi}}{1-e^{-2\pi}} \\
&= 2\pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}.
\end{aligned}$$

Hence

$$\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2} = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

□

3.(i) Suppose $A : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear map (we denote by A also its matrix). If $f \in L^1(\mathbb{R}^d)$, define $g(x) = f(Ax)$. Show that

$$\widehat{g}(\xi) = \frac{1}{|\det(A)|} \widehat{f}((A^{-1})^T \xi),$$

where $(A^{-1})^T$ is the transpose of the inverse of A.

(ii) A function $f \in L^1(\mathbb{R}^d)$ is radial if $f(x)$ depends only on $|x|$. Use (i) to show that for a radial function, the Fourier transform is radial.

(iii) Show that the result in (ii) holds also for every radial $f \in L^2(\mathbb{R}^d)$.

Proof. **(i)** We compute via the change of variables formula that

$$\begin{aligned}
\widehat{g}(\xi) &= \int_{\mathbb{R}^d} e^{-i\xi\cdot x} f(Ax)dx \\
&= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} e^{-i\xi\cdot A^{-1}x} f(x)dx \\
&= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} e^{-i(A^{-1})^T\xi\cdot x} f(x)dx \\
&= \frac{1}{|\det(A)|} \widehat{f}((A^{-1})^T\xi).
\end{aligned}$$

We also used the property of the matrix transpose that $\xi \cdot A^{-1}x = (A^{-1})^T\xi \cdot x$.
(ii) Let f be a radial L^1 function. Then $f(Ax) = f(x)$. By the first part we get that

$$\widehat{f}(\xi) = \frac{1}{|\det(A)|} \widehat{f}((A^{-1})^T\xi) = \widehat{f}((A^{-1})^T\xi),$$

where A is a rotation matrix and $\det(A) = \pm 1$. Choosing $A = (B^{-1})^T$ for some other arbitrary rotation B gives that

$$\widehat{f}(\xi) = \widehat{f}(B\xi)$$

for every rotation B. Hence $\widehat{f}$ is a radial function.
(iii) Let f be a radial L^2 function. Then $f_M = f \cdot \chi_{|x|\le M} \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is also radial. By Plancherel's theorem and the dominated convergence theorem we have that

$$\widehat{f_M} \to \widehat{f}$$

in L^2. The functions $\widehat{f_M}$ are radial by part **(ii)** so we now want to conclude that $\widehat{f}$ is also radial. Thus we want to prove that the L^2-limit of radial functions is radial.
It is not enough to prove that $\widehat{f}(B\xi) = \widehat{f}(\xi)$ for every rotation B (as L^2-functions). The problem comes from the fact that this only proves that for every rotation B, the identity $\widehat{f}(B\xi) = \widehat{f}(\xi)$ holds pointwise almost everywhere. The set of zero measure in which this identity fails might depend on B! Thus it is not immediately obvious why there is also a radial representative for our function f in its equivalence class in L^2. Let us look for a different approach.
Our original proof is based on the Lebesgue set:

$$N = \{x \in \mathbb{R}^d : x \,\text{is a Lebesgue point for}\, \widehat{f}\}.$$

Indeed, if $x, y \in N$ and $|x| = |y|$, then

$$\begin{aligned}
\widehat{f}(x) &= \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \widehat{f}(z)dz \\
&= \lim_{r\to 0} \lim_{M\to\infty} \frac{1}{|B(x,r)|} \int_{B(x,r)} \widehat{f_M}(z)dz \\
&= \lim_{r\to 0} \lim_{M\to\infty} \frac{1}{|B(x,r)|} \int_{B(y,r)} \widehat{f_M}(z)dz \\
&= \lim_{r\to 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} \widehat{f}(z)dz \\
&= \widehat{f}(y).
\end{aligned}$$

Thus f is radial in the Lebesgue set. The complement of the Lebesgue set is of measure zero, so we can redefine f in the complement so that it is radial everywhere.
Another proof for the same fact is to choose a subsequence of $\widehat{f_M}$ that converges to $\widehat{f}$ pointwise almost everywhere. This is possible as proven in the real analysis course. Since $\widehat{f}$ is almost everywhere a pointwise limit of radial functions, it must have a radial representative in L^2. □

4. Show that if E is a fundamental solution of the differential operator (with constant coefficients) $P(\partial)$, then $E + H$ is also a fundamental solution, if $H \in \mathcal{S}'(\mathbb{R}^d)$ satisfies

$P(\partial)H = 0$.Verify that actually all fundamental solutions of P are obtained by this manner.

Proof. As the considered differential operator is linear, we see that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle P(\partial)(E+H), \varphi\rangle = \langle P(\partial)E, \varphi\rangle + \langle P(\partial)H, \varphi\rangle = \langle \delta_0, \varphi\rangle + \langle 0, \varphi\rangle = \langle \delta_0, \varphi\rangle.$$

Next assume that E_1, E_2 are two fundamental solutions. Then $E_2 = E_1 + (E_2 - E_1)$ and we have for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\langle P(\partial)(E_2 - E_1), \varphi\rangle = \langle P(\partial)E_2, \varphi\rangle - \langle P(\partial)E_1, \varphi\rangle = \langle \delta_0, \varphi\rangle - \langle \delta_0, \varphi\rangle = 0.$$

Therefore all fundamental solutions can be obtained from one by adding $H \in \mathcal{S}'(\mathbb{R}^d)$ that satisfies $P(\partial)H = 0$. □

5. The Weierstrass type function

$$f(x) := \sum_{n=1}^{\infty} 2^{-n/2} \cos(2^n x)$$

can be shown not to be differentiable at any point. Show in any case that in the sense of distributions we have

$$f'(x) = -\sum_{n=1}^{\infty} 2^{n/2} \sin(2^n x) \qquad !$$

Proof. Define

$$f_N(x) = \sum_{n=1}^{N} 2^{-n/2} \cos(2^n x).$$

Then the f_N are continuous functions and $f_N \to f$ uniformly as $n \to \infty$. The uniform convergence follows from

$$|f(x) - f_N(x)| \leq \sum_{n=N+1}^{\infty} 2^{-n/2} \to 0.$$

Thus $f_N \to f$ in the sense if distributions, which implies that $f_N' \to f'$ in the sense of distributions. Thus

$$f_N'(x) = -\sum_{n=1}^{N} 2^n 2^{-n/2} \sin(2^n x) = -\sum_{n=1}^{N} 2^{n/2} \sin(2^n x) \to -\sum_{n=1}^{\infty} 2^{n/2} \sin(2^n x) = f'(x)$$

in the sense of distributions, which is what we wanted to prove. □

6. Let $A = \{(x,y) : x > 0, y > 0\} \cup \{(x,y) : x < 0, y < 0\} \subset \mathbb{R}^2$.

Show that the characteristic function χ_A is a fundamental solution for the differential operator $P_1(\partial) = \frac{1}{2}\partial_1\partial_2$.

Proof. We need to check that in the sense of distributions the following identity holds:

$$P_1(\partial)\chi_A = \delta_0.$$

This is just a simple calculation as follows:

$$\begin{aligned}
\langle P_1(\partial)\chi_A, g\rangle &= \left\langle \chi_A, \frac{1}{2}g_{xy}\right\rangle \\
&= \frac{1}{2}\iint_A g_{xy}(x,y)dxdy \\
&= \frac{1}{2}\int_0^\infty\int_0^\infty g_{xy}(x,y)dxdy + \frac{1}{2}\int_{-\infty}^0\int_{-\infty}^0 g_{xy}(x,y)dxdy \\
&= \frac{1}{2}\int_0^\infty -g_y(0,y)dy + \frac{1}{2}\int_{-\infty}^0 g_y(0,y)dy \\
&= \frac{g(0,0)}{2} + \frac{g(0,0)}{2} \\
&= \langle \delta_0, g\rangle, \quad \text{for all} \quad g \in \mathcal{S}(\mathbb{R}^2).
\end{aligned}$$

□

7. (i) If $0 < \gamma < d$, show that the function

$$f_\gamma(x) = \frac{1}{|x|^\gamma}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

determines a tempered distribution, by writing it as a sum of two functions, one belonging to $L^1(\mathbb{R}^d)$ and the other to $L^p(\mathbb{R}^d)$ for a suitable $p > 1$. If $\gamma > d/2$, show that one can choose $p = 2$.

(ii) Prove that

$$\widehat{f_\gamma}(\xi) = c(d,\gamma)\frac{1}{|x|^{d-\gamma}}$$

for some constant $c(d,\gamma)$.

Proof. (i) If $0 < \gamma < \delta$, we write that

$$f_\gamma(x) = \chi_{|x|<1}\frac{1}{|x|^\gamma} + \chi_{|x|\geq 1}\frac{1}{|x|^\gamma}.$$

The first part is in L^1 since by a polar change of variables we find that

$$\int_{|x|<1} \frac{1}{|x|^\gamma} dx = C\int_0^1 \frac{1}{r^\gamma} r^{d-1} dr,$$

and since the exponent $d - \gamma - 1 > -1$, this integral is finite. For the second part to be in L^p, the integral

$$\int_{|x|\geq 1} \frac{1}{|x|^{p\gamma}} dx = C\int_1^\infty \frac{1}{r^{p\gamma}} r^{d-1} dr$$

must be finite. Thus we get the inequality $d - p\gamma - 1 < -1$, which is true when $p > d/\gamma$. If $\gamma > d/2$, we see from this that one may choose $p = 2$. Thus in general f_γ is a sum of an L^1-function and an L^p-function, and hence defines a tempered distribution on $\mathbb{R}^d$.
(ii) Case $\gamma > d/2$. In this case f_γ is a sum of an L^1-function and a L^2-function. Thus we find that the Fourier transform $\widehat{f_\gamma}$ is also a function (it is a sum of L^∞-function and a L^2-function). Note that the function f_γ has the property that

$$f_\gamma(tx) = t^{-\gamma} f_\gamma(x) \text{ for every } t > 0 \text{ and every } x.$$

Let $g \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\begin{aligned}
\Big\langle \widehat{f_\gamma}(x/t), g(x) \Big\rangle &= \Big\langle \widehat{f_\gamma}(x), t^d g(tx) \Big\rangle \\
&= \Big\langle f_\gamma(x), \widehat{t^d g(tx)} \Big\rangle \\
&= \Big\langle f_\gamma(x), \widehat{g}(x/t) \Big\rangle \\
&= \Big\langle t^d f_\gamma(tx), \widehat{g}(x) \Big\rangle \\
&= t^{d-\gamma}\Big\langle f_\gamma(x), \widehat{g}(x) \Big\rangle \\
&= \Big\langle t^{d-\gamma}\widehat{f_\gamma}(x), g(x) \Big\rangle.
\end{aligned}$$

Thus $\widehat{f_\gamma}(x/t) = t^{d-\gamma}\widehat{f_\gamma}(x)$ in the distributional sense. The above formula can also be generalized to functions g in $L^1 \cap L^2$ by approximation. We want to conclude that there is a representation of the function $\widehat{f_\gamma}$ (which is defined almost everywhere) that also satisfies the identity

$$\widehat{f_\gamma}(x/t) = t^{d-\gamma}\widehat{f_\gamma}(x) \tag{1}$$

at every point $x \in \mathbb{R}^d$. For this we again use the Lebesgue set

$$N = \{x \in \mathbb{R}^d : x \text{ is a Lebesgue point for } \widehat{f_\gamma}\}.$$

If $x \in N$ and $x/t \in N$, then a computation similar to the one in exercise 3 gives that

$$\begin{aligned}
\widehat{f_\gamma}(x/t) &= \lim_{r\to 0} \frac{1}{|B(x/t,r)|} \int_{B(x/t,r)} \widehat{f_\gamma}(z)dz \\
&= \lim_{r\to 0} \frac{1}{|B(x/t,r)|} \int_{B(x/t,r)} t^{-d}\widehat{f_\gamma}(z/t)dz \\
&= \lim_{r\to 0} \frac{1}{|B(x/t,r)|} \Big\langle t^{-d}\widehat{f_\gamma}(z/t), \chi_{B(x,tr)}(z)\Big\rangle \\
&= \lim_{r\to 0} \frac{t^d}{|B(x,tr)|} \Big\langle t^{-\gamma}\widehat{f_\gamma}(z), \chi_{B(x,tr)}(z)\Big\rangle \\
&= \lim_{r\to 0} \frac{t^{d-\gamma}}{|B(x,tr)|} \int_{B(x,tr)} \widehat{f_\gamma}(z)dz \\
&= t^{d-\gamma}\widehat{f_\gamma}(x).
\end{aligned}$$

The computation is valid since $\chi_{B(x,tr)}(z) \in L^1 \cap L^2$. Since the Lebesgue set N contains almost every point in $\mathbb{R}^d$, we can redefine f outside of N so that it satisfies the identity $\widehat{f_\gamma}(x/t) = t^{d-\gamma}\widehat{f_\gamma}(x)$ everywhere. Now we use this formula to compute that for every ξ we have

$$\widehat{f_\gamma}(\xi) = \widehat{f_\gamma}(\xi/|\xi|)\frac{1}{|\xi|^{d-\gamma}}.$$

The point $\xi/|\xi|$ is on the unit sphere and by exercise 3 our function $\widehat{f_\gamma}$ is radial (by combining the arguments we can actually choose a representative that is both radial and satisfies the identity (1)). Thus $\widehat{f_\gamma}(\xi/|\xi|)$ does not depend on the choice of ξ. Hence we can denote it by a constant $c(d,\gamma)$ and we get that

$$\widehat{f_\gamma}(\xi) = \frac{c(d,\gamma)}{|\xi|^{d-\gamma}}$$

as wanted.

Remark. *It was very important to choose the correct representative of f, as the unit sphere has zero measure in $\mathbb{R}^d$ and thus the value of $c(d,\gamma)$ would otherwise depend on the representative chosen, which would make no sense.*

Case $\gamma < d/2$. In this case we have that $d - \gamma > d/2$. From the previous case it follows that

$$\widehat{f_{d-\gamma}}(\xi) = \frac{c(d,d-\gamma)}{|\xi|^\gamma} = c(d,d-\gamma)f_\gamma(\xi).$$

This Fourier transform is taken in the sense of distributions. We can take the inverse Fourier transform of both sides (again in the sense of distributions) to get that

$$f_{d-\gamma}(x) = c(d, d-\gamma)\mathcal{F}^{-1} f_\gamma(x) = c(d, d-\gamma)(2\pi)^{-d} \widehat{f_\gamma}(x).$$

The constant $c(d, d-\gamma)$ cannot be zero, so we get that

$$\widehat{f_\gamma}(\xi) = \frac{c(d,\gamma)}{|\xi|^{d-\gamma}} \quad \text{where} \quad c(d,\gamma) = \frac{(2\pi)^d}{c(d, d-\gamma)}$$

as wanted.
Case $\gamma = d/2$. Let us investigate what happens when $\gamma \to d/2$. By dominated convergence theorem we have that

$$\langle f_\gamma, g \rangle = \int_{\mathbb{R}^d} \frac{g(x)}{|x|^\gamma} dx \to \int_{\mathbb{R}^d} \frac{g(x)}{|x|^{d/2}} = \langle f_{d/2}, g \rangle.$$

This shows that $f_\gamma \to f_{d/2}$ in $\mathcal{S}'(\mathbb{R}^d)$. Thus must also have that $\widehat{f_\gamma} \to \widehat{f_{d/2}}$ in $\mathcal{S}'(\mathbb{R}^d)$ as $\gamma \to d/2$, since we already know that the distribution $\widehat{f_{d/2}}$ is well-defined. Now let us choose a subsequence γ_k so that the numbers $c(d, \gamma_k)$ converge either to a real number c or to $\pm\infty$. We can compute again by dominated convergence theorem that

$$\begin{aligned} \lim_{k\to\infty} \left\langle \widehat{f_{\gamma_k}}, g \right\rangle &= \lim_{k\to\infty} \int_{\mathbb{R}^d} \frac{c(d,\gamma_k)}{|x|^{d-\gamma_k}} g(x) dx \\ &= \left(\lim_{k\to\infty} c(d,\gamma_k) \right) \int_{\mathbb{R}^d} \frac{1}{|x|^{d/2}} g(x) dx. \end{aligned}$$

We know that the limit of the right hand side cannot be $\pm\infty$ since the $\widehat{f_{\gamma_k}}$ must have a distributional limit (which is $\widehat{f_{d/2}}$). Thus the $c(d, \gamma_k)$ must converge to a real number c. This gives that

$$\left\langle \widehat{f_{d/2}}, g \right\rangle = \lim_{k\to\infty} \left\langle \widehat{f_{\gamma_k}}, g \right\rangle = \langle c/|\xi|^{d/2}, g \rangle.$$

If follows that $\widehat{f_{d/2}} = c/|\xi|^{d/2}$ as wanted.

Remark. *It is in fact possible to determine the constants $c(d,\gamma)$ exactly with help of the function $e^{-|x|^2}$, whose Fourier transform is easy enough to find. The value turns out to be*

$$c(d,\gamma) = \frac{\pi^{d/2} 2^{d-\gamma} \Gamma(\frac{d-\gamma}{2})}{\Gamma(\gamma/2)},$$

where Γ is the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

$\square$

8. Try to find a fundamental solution to Δ^2 in $\mathbb{R}^3$.

Proof. We try to find the distribution E with

$$\Delta^2 E = \delta_0.$$

Suppose that E is such a fundamental solution. Taking the Fourier transform and recalling exercise 6 from exercise set 2, we see that this is equivalent to

$$|\xi|^4 \widehat{E} = 1.$$

Here we run into a problem: the function $\xi \mapsto |\xi|^{-4}$ is not locally integrable and does not define a tempered distribution in an obvious way.
However, we can get past this problem by considering a suitable singular integral. Denote by λ the operator with

$$\langle \lambda, g \rangle := \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{g(x) - g(0)}{|x|^4} dx,$$

if the limit exists. We claim that the limit exists for any $g \in \mathcal{S}(\mathbb{R}^3)$ and that $\lambda \in \mathcal{S}'(\mathbb{R}^3)$. We split the integral into two parts:

$$\int_{|x|>\epsilon} \frac{g(x) - g(0)}{|x|^4} dx = \int_{|x|\geq 1} \frac{g(x) - g(0)}{|x|^4} dx + \int_{\epsilon<|x|<1} \frac{g(x) - g(0)}{|x|^4} dx.$$

We can estimate the first part as

$$\left| \int_{|x|\geq 1} \frac{g(x) - g(0)}{|x|^4} dx \right| \leq \int_{|x|\geq 1} \frac{2p_0(g)}{|x|^4} dx = Dp_0(g),$$

where D is some independent constant. For the second part, we know that the second order partial derivatives of g are bounded by $p_2(g)$, so we have the following Taylor approximation near zero:

$$g(x) = g(0) + \nabla g(0) \cdot x + |x|^2 A(x),$$

where A is a function bounded by $C_0 p_2(g)$ for some constant C_0 in the neighbourhood of zero. Then we can use the fact that $\nabla g(0) \cdot x$ is odd to estimate

$$\left| \int_{\epsilon<|x|<1} \frac{g(x) - g(0)}{|x|^4} dx \right| = \left| \int_{\epsilon<|x|<1} \left(\frac{\nabla g(0) \cdot x}{|x|^4} + \frac{A(x)}{|x|^2} \right) dx \right|$$

$$
\begin{aligned}
&= \left| \int_{\epsilon<|x|<1} \frac{A(x)}{|x|^2} dx \right| \\
&\to \left| \int_{|x|<1} \frac{A(x)}{|x|^2} dx \right| \\
&\leq \int_{|x|<1} \frac{C_0 p_2(g)}{|x|^2} dx \\
&= C_1 p_2(g),
\end{aligned}
$$

where we used the fact that $x \mapsto |x|^{-2}$ is locally integrable. As λ is linear and bounded by p_2, we see that $\lambda \in \mathcal{S}'(\mathbb{R}^3)$.

We observe that $\langle \lambda, |x|^4 g \rangle = \langle 1, g \rangle$, so now we need to find a tempered distribution E with $\widehat{E} = \lambda$. Recall from exercise 4 of the previous exercise set that $\langle \mathcal{F}^{-1}T, g \rangle = \langle T, \mathcal{F}^{-1}g \rangle$ for any $T \in \mathcal{S}'(\mathbb{R}^3)$. So we will now use the inversion formula:

$$
\begin{aligned}
\langle E, g \rangle = \langle \mathcal{F}^{-1}\lambda, g \rangle &= \langle \lambda, \mathcal{F}^{-1} g \rangle \\
&= \lim_{\epsilon \to 0^+} \int_{|\xi|>\epsilon} \frac{\mathcal{F}^{-1}g(\xi) - \mathcal{F}^{-1}g(0)}{|\xi|^4} d\xi \\
&= \lim_{\epsilon \to 0^+} \int_{|\xi|>\epsilon} |\xi|^{-4} \left(\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) e^{i\xi\cdot x} dx - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) dx \right) d\xi \\
&= \frac{1}{(2\pi)^3} \lim_{\epsilon \to 0^+} \int_{|\xi|>\epsilon} \int_{\mathbb{R}^3} |\xi|^{-4} g(x) (e^{i\xi\cdot x} - 1) dx d\xi \\
&= \frac{1}{(2\pi)^3} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^3} g(x) \int_{|\xi|>\epsilon} |\xi|^{-4} (e^{i\xi\cdot x} - 1) d\xi dx,
\end{aligned}
$$

where in last equality we used Fubini's theorem since the integrand is bounded in absolute value by $2|g(x)||\xi|^{-4}$ and hence it is absolutely integrable.

We will show that

$$
\int_{|\xi|>\epsilon} |\xi|^{-4} (e^{i\xi\cdot x} - 1) d\xi \leq C' |x|
$$

for some constant C'. This would allow us to use the dominated convergence theorem because $|x|g$ is an integrable function. For this, we will pass to the polar coordinates. For $x \neq 0$ we have that

$$
\begin{aligned}
\int_{|\xi|>\epsilon} |\xi|^{-4} (e^{i\xi\cdot x} - 1) d\xi &= \int_{\epsilon}^{\infty} \int_{S^2} r^2 r^{-4} (e^{iru\cdot x} - 1) dS(u) dr \\
&= \int_{\epsilon}^{\infty} \int_{S^2} r^{-2} (e^{iu\cdot rx} - 1) dS(u) dr
\end{aligned}
$$

$$= \int_{\epsilon|x|}^{\infty} \int_{S^2} (y/|x|)^{-2} (e^{iu\cdot(y/|x|)x} - 1) dS(u) \frac{dy}{|x|}$$
$$= |x| \int_{\epsilon|x|}^{\infty} y^{-2} \int_{S^2} (e^{iyu\cdot(x/|x|)} - 1) dS(u) dy,$$

where we made a change of variables $r|x| = y$. Now we observe that $x/|x|$ is a unit vector. The value of the inner integral is independent of x as we are integrating over the whole unit sphere (we can see this by taking a rotation).
If we split this integral and use the symmetry, we see that the limit $\epsilon \to 0^+$ exists. In particular, the integral is bounded by $C'|x|$ and has a limit $C|x|$ for some constant C. Using dominated convergence theorem, we finally obtain

$$\langle E, g \rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) \lim_{\epsilon \to 0^+} \int_{|\xi|>\epsilon} |\xi|^{-4} (e^{i\xi\cdot x} - 1) d\xi dx = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} g(x) C|x| dx.$$

This means that the fundamental solution of Δ^2 is of the form $C|x|$. $\square$