FOURIER ANALYSIS II. (spring 2020)

6. (EXTRA) EXERCISES (Thursday 7.5)

NOTE: return these exercises at the latest as pdf and by e-mail (clear hand-writing, or typed) to

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They should be returned at the latest during the day of the 'exercise class'. The course will be evaluated based on the returned exercises.

These exercises are 'extras', and you may use them to increase your total saldo of points.

1. Compute the Fourier transform of the function $f : \mathbf{R} \to \mathbf{C}$, where

$$f(x) = \begin{cases} 0, & \text{for } x < 0, \\ e^{-x} & \text{for } x \geq 0. \end{cases}$$

2. Show that if $f, g \in \mathcal{S}(\mathbf{R})$ are test functions, then also their product is: $fg \in \mathcal{S}(\mathbf{R})$.

3. Let a be a real constant. Show that the equation

$$u''(x) + au'(x) - u(x) = 0 \quad \text{for all} \quad x \in \mathbf{R}.$$

has no other solutions $u \in \mathcal{S}'(\mathbf{R})$ than the trivial solution $u \equiv 0$.

4. Let $f : \mathbf{R} \to \mathbf{C}$ be a piecewise continuously differentiable function[1]. Assume also that f and f' are bounded[2] as $x \to \infty$. Let us denote by $f'(x)$ the pointwise derivative of f, which now exists almost everywhere. Show that the distributional derivative $\frac{df}{dx}$ can be written as

$$\frac{df}{dx} = f'(x) + \sum_{j=1}^{k} \big(f(a_j+)) - f(a_j-)\big)\delta_{a_j}$$

(here of course $f'(x)$ stands for the distribution $T_{f'(x)}$).

5. Determine the distributional derivatives $\frac{df}{dx}$ and $\frac{d^2 f}{d^2 x}$ of the function $f(x) := \max(0, 1 - |x|)$.

[Hint: (ii): do it directly or apply the previous exercise]

[1] i.e. there are only finitely many points (or none) $a_1 < a_2 < \cdots < a_k$ so that on every open interval of the set $\mathbf{R} \setminus \{a_1, \ldots a_k\}$ the function f is in C^1, and the right or left limits

$$f(a_j+) := \lim_{t \to 0^+} f(a_j + t) \quad \text{ja} \quad f'(a_j+) := \lim_{t \to 0^+} f'(a_j + t)$$

exits, as well us the limits $f(a_j-) := \lim_{t \to 0^+} f(a_j - t)$ ja $f'(a_j-) := \lim_{t \to 0^+} f'(a_j - t)$

[2] it would be enough for them to grow polynomially

6. Let $f, g \in C(\mathbf{R})$ be arbitrary. Show that the function

$$u(x,t) := f(x+t) + g(x-t)$$

satisfies the wave equation

$$\Big(\frac{d}{dt}\Big)^2 u = \Big(\frac{d}{dx}\Big)^2 u$$

on $\mathbf{R}^2$, where the derivatives are taken in the sense of distributions – thus d'Alembert's formula works fine in great generality!

[Hint: let $\varphi \in S(\mathbf{R}^2)$ be an arbitrary test function. Concerning the term $f(x+t)$ by the definition of the distributional derivative you need to show that

$$\int_{\mathbf{R}^2} f(x+t)\big(\varphi_{xx}(x,t) - \varphi_{tt}(x,t)\big)dtdx = 0.$$

To do that one possibility is to make a change of variables $x+t = z$, $x - t = w$, and express the integral and the operator $\big(\frac{d}{dt}\big)^2 - \big(\frac{d}{dx}\big)^2$ in terms of these variables.]

7. Prove a stronger form of Theorem 13.8 of lectures: show that if $f \in \mathcal{S}(\mathbf{R})$ satisfies $\int_{\mathbf{R}} |f(x)|^2 dx = 1$, then

$$\Big(\int_{\mathbf{R}} (x-A)^2 |f(x)|^2 dx\Big)\Big(\int_{\mathbf{R}} (\xi - B)^2 |\widehat{f}(\xi)|^2 d\xi\Big) \geq \frac{\pi}{2}$$

for all constants $A, B \in R$.

[Hint: apply Thm 13.8 on $e^{iax} f(x+b)$ with suitably chosen a, b.]

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8[*3]

(i) Show that the Fourier-transform of the function

$$f(x) = \begin{cases} 0, & \text{for } |x| \geq 1, \\ \log(1/|x|) & \text{for } |x| < 1. \end{cases}$$

can be written in the form (for $\xi \neq 0$)

$$\widehat{f}(\xi) = C|\xi|^{-1} \int_0^{|\xi|} \frac{\sin(t)}{t} dt.$$

(ii) Use part (i) to show that in (one-dimensional) Theorem 13.15 of the Lectures one may not decrease the constant 1/2 in the condition $s > 1/2$ In other words if $s < 1/2$, then $W^{s,2}(\mathbf{R}) \not\subset C(\mathbf{R})$.

[3]These $*$-exercises are extras for afficinadoes, not required to get full points from exercises