FOURIER ANALYSIS II, Spring 2020

Lecturer: Eero Saksman

Timetable of the lectures:

Your exercise saldo (from which you can estimate your tentative grade) is now posted on the web, as well as the EXTRA exercise set 6. Then you can decide whether you still want to try to increase your saldo by returning also the set 6 (it needs to be returned at the latest on Thursday 7.5.

Update due to the coronavirus epidemic: As mentioned in the e-mail on 13.3, at the moment the contact lectures are cancelled due to the policy of the department and university. Right now we will continue the course as self study. **The course will be evaluated by the returned exercises only. The lowest number of accepted returns to pass the course is 17 (out of total number 35). There will be 1 extra exercise class (repeating basic material) for those who want to increase the number of their approved exercises – these are not needed for obtaining maximal number of points**. Exercise sets should be returned as a pdf by e-mail to the course assistant **at the latest during the day of the 'exercise-class'**. I will further investigate the possibility, whether we could have one streamed session per week (2-4 hours) when I would be available to try to answer questions you are sending during the session. Whether this works our we will see most likely next week - right now I'm still in quarantine.
You will find inside the Finnish lecture notes an English translation of the material you are expected to study yourself. Returned exercises may be handwritten (clearly enough) or LaTeX:ified. Further instructions will possibly appear as soon as things clarify. I'm of course very sorry that the lectures cannot be continued as originally planned.
It seems that we will be done with the 'lectures' on 8.4. However, there will be 1 extra exercise class(repeating basic material) for those who want to increase the number of their approved exercises – these are not needed for obtaining maximal number of points

The 3 weekly lectures are on Mondays (C122), Tuesdays (C122), and Thursdays (B322) at 10-12. See for the weekly exercise class (Thursdays 14-16 C123) below. During some weeks there are less lectures. Here is the preliminary schedule (−− means no lecture on that day) **PLEASE,**

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Week	Monday	Tuesday	Thursday
11	9.3.	10.3	12.3.
12	16.3.	17.3	19.3.
13	23.3.	24.3.	26.3.
14	30.3.	31.3.	2.4.
15	6.4.	8.4.	–
16			16.4.

Exercise classes:

Exercise classes are given by Stefanos Lappas (`stefanos.lappas@helsinki.fi`), They are held on **Thursdays 14-16** in room C123. By exercises alone one can gather extra points for the exam (8/24 extra points at maximum). The exercises will appear at the webpage in section 'Materials'.

Blog

`Abreviations` [StSh] refers to E.M. Stein & R. Shakarchi: *Fourier Analysis, An Introduction*, Princeton University Press, 2003. [LN] refers to the Finnish-English lecture notes..

Monday 9.3 Introduction, heuristics of the Fourier transform starting from L-periodic Fourier series and letting $L \to \infty$. Rigorous definition and basic properties of the Fourier transform of $L^1(\mathbb{R}^d)$-functions. Derivative of the Fourier transform, Fourier- transform of the derivative. Behaviour under scaling and translation. Riemann-Lebesgue lemma. Fourier transform of a convolution. $\big($ [LN] pp. 79–83.$\big)$

Tuesday 10.3 Duality pairing ('multiplication formula') of the Fourier-transform. Heuristics of the inverse formula. Fourier-transfrom of Gaussian functions. Rigorous proof of the inversion formula assuming that $f, \widehat{f} \in L^1(\mathbb{R}^d)$. $\big($ [LN] pp. 84–87.$\big)$

Thursday 12.3 Consequences of the inverse formula. Fourier-transfrom of Gaussian functions. Some examples of Fourier transforms. Extensive discussion of background and motivation related to our goal to enlarge the Fourier transform to a larges class (i.e. to distributions). Schwarz test functions, multi-index notation $\big($ [LN] pp. 88–89.$\big)$

Monday 16.3 Definition of Schwarz test functions recalled. Basic properties of the Schwarz test functions. Especially, invariance under Fourier transform, differentiation and multiplication by polynomials. Convolutions and inverse Fourier transform for test functions. $\big($ [LN] pp. 88–92.$\big)$

Tuesday17.3 L^2-theory of the Fourier transform (up to p. 97) ([LN] pp. 93–97.)

Thursday19.3 L^2-theory of the Fourier transform continued. Idea of interpolation. Extending operator from a dense subspace (Theorem 11.1). Statement of the Riesz.Thorin interpolation theorem (Teorem11.2). Application to the Fourier transform: Hausedorff-Young inequality (Corollary 11.3). ([LN] pp. 98–102.)

Monday 23.3 Proof of Riesz-Thorin interpolation theorem. ([LN] pp. 102–108.)

Tuesday 24.3 Speculation on extending the Fourier transfrom. Seminorms on $\mathcal{S}$, and metric on $\mathcal{S}$. Completeness of the metric. Characterization of continuity of linear functionals or operators on $\mathcal{S}$. Continuity of multiplication by polynomials, differentiation, and taking Fourier transform. ([LN] pp. 109–113.)

Thursday 26.3 Motivation of the definition of a distribution. The linear map (distribution) T_f defined by (e.g.) an integrable function. Definition of (tempered) distributions. An function in $L^p(\mathbb{R}^d)$ defines the distribution T_f. Delta functions, and other examples. Multiplying a distribution by a smooth function. ([LN] pp. 113–117.)

Monday 30.3 Differentiation of distributions. Fourier transform of distributions. Examples. ([LN] pp. 117–123.)

Tuesday 31.3 Principal value integral as a distribution. Hilbert transform. Fourier transform of the principal value integral. ([LN] pp. 123–126.)

Thursday 2.4 Convolution of distributions. ([LN] pp. 127–129.)

Monday 6.4 Poisson summation formula. Dirac combs and their Fourier transforms. Applications to the heat equation.([LN] pp. 130–134.)

Tuesday 7.4 Fundamental solutions. ([LN] pp. 135–139.)

Thursday 16.4 Uncertainty principle. Sobolev spaces on $\mathbb{R}^d$ (THE LAST LECTURE) ([LN] pp. 139–145.)

Content

Fourier analysis is a central tool in many areas of mathematics, including PDE:s, harmonic analysis, analytic number theory, mathematical physics, probability, etc... The aim of the course 'Fourier analysis II' is to bring further the material of the course 'Fourer analysis I' and give basic and workable knowledge on Fourier transform on R^n. Moreover, we provide an introduction to the theory of distributions (generalised functions).

Lectures will follow somewhat freely the Finnish language notes[1] included on the webpage section 'Materials' .The actual content of the lectures can be seen by following the BLOG of the course below, where also corresponding pages of the following English text books will be referenced:

E.M. Stein & R. Shakarchi: *Fourier Analysis, An Introduction*, Princeton University Press, 2003

R. Strichartz: *A guide to distribution theory and Fourier transforms*, CRC Press. 1994. 2nd edition. World Scientific. 2003

If needed, some more material will be provided during the first lectures. For the interested students the following two books are useful companions:

- L. Grafakos: *Classical Fourier Analysis*, Springer, 2008.
- Rudin, W.: *Real and Complex Analysis*, McGraw-Hill, Third ed. 1987;
  Rudin, W.: *Functional analysis*, McGraw-Hill, Second ed. 1990.

[1]The Finnish language lecture notes (covering both Fourier analysis I and II) are based on those produced by Kari Astala on fall 2012, that have been somewhat modified by later lecturers including the present one.