Algebra II

Preliminary exercises 13

1. Prove that $\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}]$, that is, every number of the form

$$\frac{a + b\sqrt{2}}{c + d\sqrt{2}},$$

where $a, b, c, d \in \mathbb{Q}$, can be written in the form $e + f\sqrt{2}$, where $e, f \in \mathbb{Q}$.

2. Suppose that L is an extension of a field K and $a, b \in L$. Prove that $K(a, b) = K(a)(b)$.

3. Suppose $n \in \{2, 3, ...\}$. Prove that the polynomial $X^n - 2$ is irreducible in $\mathbb{Q}[X]$. Deduce that it is the minimal polynomial of the number $\sqrt[n]{2}$ with respect to $\mathbb{Q}$.

4. Determine $\min(\mathbb{Q}, \sqrt{2}i)$.

5. Let R be a ring. The set

$$R[X_1, ..., X_n]$$

of polynomials having n indeterminates with coefficients in R is an R-algebra. Sketch the proof for the following result:
Let R be a ring and A some associative, commutative and unitary R-algebra. Let $(x_1, ..., x_n)$ be a sequence of elements of A. Then there exists a unique R-algebra homomorphism $\varphi\colon R[X_1, ..., X_n] \to A$ for which $\varphi(X_i) = x_i$ for every i.

The map φ is called the *evaluation homomorphism* associated with the elements $x_1, ..., x_n$. (In case $A = R, n = 1$, and given $a \in R, f \in R[X]$, then $\varphi(f)$ means just that we substitute a in the polynomial f.)