Algebra II

Exercise 13 (6.5.2020)

1. See Example 13.5 (7.5). Represent the numbers

$$\frac{1}{\sqrt[3]{2}} \text{ ja } \frac{\sqrt[3]{2}-1}{\sqrt[3]{2}-\sqrt[3]{4}}$$

in the form $a+b\sqrt[3]{2}+c\sqrt[3]{4}$, where $a,b,c\in\mathbb{Q}$.

2. Prove Proposition 13.7 (7.7): Let L be a finite extension of a field K. Then L is finitely generated and algebraic with respect to K.

3. a) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(i)$ are isomorphic as $\mathbb{Q}$-vector spaces, but not as fields.
b) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic as fields.

4. a) Prove: If L is an extension of a field K and $A\subset L$ is a set of elements, which are algebraic with respect to K, then the extension $K(A)/K$ is algebraic.
b) Let

$$A=\{2^{\frac{1}{n}} \mid n\in\mathbb{N}\}.$$

Prove that $\mathbb{Q}(A)$ is an infinite algebraic extension of the field $\mathbb{Q}$. Does the extension $\mathbb{Q}(A)/\mathbb{Q}$ have a finite generating set?

5. a) Let R be a ring and $b\in R$. Prove that the map

$$\tau_b\colon R[X]\to R[X],$$

where

$$\sum_i a_iX^i\mapsto\sum_i a_i(X+b)^i,$$

is a ring isomorphism. Deduce that $f\in R[X]$ is irreducible, if and only if $\tau_b(f)$ is irreducible.
b) Let p be a prime number. Prove that $X^p-1=(X-1)g$, where $g\in\mathbb{Z}[X]$ is irreducible with respect to the field $\mathbb{Q}$.
[Hint: Consider the polynomial $\tau_1(X^p-1)$ and use Eisenstein's criterion.]

6. In the following cases, determine the degree of the subextension $K(A)$ generated by A of the extension $\mathbb{C}/\mathbb{Q}$:

a) $A = \{\sqrt{2}, i\}$

b) $A = \{e^{2\pi i/5}\}$

c) $A = \{i, e^{\pi i/3}\}$ [Hint: Prove first that $\mathbb{Q}(i, e^{\pi i/3}) = \mathbb{Q}(i, \sqrt{3})$].